New Isotopic Spin Space
and Classification of Elementary Particles.

P. HILLION and J.-P. VIGIER

Institut H. Poincaré - Paris

Summary. — We start from the new idea that isotopic spin space is invariant under the group of three dimensional complex conjugate rotations and justify this assumption on the basis of a classical model of extended particles described by relativistic rotators. We then express in terms of Euler parameters (a relativistic generalization of real three dimensional Euler angles) the infinitesimal operators of the Lie algebra associated with this group and obtain, in the general space of the corresponding eigenfunctions, finite dimensional sub-spaces associated with irreducible representations of the complex rotation group. In each sub-space one then readily defines a set of « isotopic » or « internal state vectors » corresponding to different types of elementary particles so that each particle family is connected with an irreducible representation of this group. In this scheme isotopic strangeness and fermionic numbers correspond to certain infinitesimal rotation operators and one justifies (with new qualitative aspects) the empirical Nishijima–Gell-Mann classification of elementary particles.

Introduction.

During the last period many types of isotopic spin spaces have been proposed [1] all starting from two basic ideas:

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[1] See the paper by D'ESPAGNAT and PRENTKI in *Progress in Elementary Particle and Cosmic Ray Physics*, vol. **4** (Amsterdam, 1958).

a) Elementary particles can be described by δ-functions in a Minkowski space;

b) The new degrees of freedom, which eventually correspond to the new quantum numbers (iso-spin, strangeness ...) are necessarily associated to motions in an abstract isotopic spin-space different from physical space-time.

It was then successively assumed to have:

— three dimensions, Euclidean metric and to be invariant under the group of real three-dimensional rotations;

— four dimensions with Euclidean metric and to be invariant under the group of real four-dimensional rotations;

— four dimensions with Lorentz metric and to be invariant under the Lorentz group.

In this paper, our starting point will be different: we think that elementary particles have an internal structure and we intend to explain isotopic spin, strangeness, baryonic numbers and ordinary spin by certain movements of this structure in ordinary space-time.

But before we start, we want to emphasize the fact that the results obtained here would also come true if one directly postulated an abstract isotopic spin, invariant under the three-dimensional complex rotation group. This could be justified by the consideration of the isomorphism between this group and the Lorentz one (*).

Analysing with Professor BOHM ([2]) the structure of classical extended particles in space-time, we were led to compare them with hyperspherical relativistic rotators. Such a model implies that, after quantization ([3]), the new degree of freedom correspond to certain symmetry properties of internal waves associated with this structure. This leads to the introduction of a new type of isotopic spin space which turns out to be a manifold invariant under the group of three-dimensional complex rotations. Starting from this new qualitative step and following complex usual quantization procedure we are naturally led to:

— describe these complex rotations and the corresponding infinitesimal operators in terms of suitable parameters which turn out to be the general-

(*) *Note added in proof.* - Now we have shown that the two groups were only locally isomorphic and that the representations of R_3^* are pseudo-unitary because of the introduction of two types of complex units i and j.

([2]) D. BOHM, P. HILLION, T. TAKABAYASI and J.-P. VIGIER: *Relativistic rotator and bilocal theory, Progr. Theor. Phys.*, **23**, 496 (1960).

([3]) D. BOHM, P. HILLION and J.-P. VIGIER: *Internal quantum states of the hyperspherical relativistic rotator*, to be published in *Progr. Theor. Phys.*

ization to space-time of the usual Euler angles. This will be developped in Section **1**;

— study the corresponding eigenfunctions located in a Hilbert space invariant under this group. We utilize the fact that this space splits into sub-spaces which correspond to irreducible representations of the group and define « isotopic vectors » or « internal state vectors » which we propose to associate with elementary particles. This will be done in Section **2**.

In Section **3**, finally, we shall introduce infinitesimal rotation operators in these sub-spaces which when applied to the preceding « internal state vectors » yield eigenvalues which correspond to the isotopic spin, strangeness and Fermionic number of the Nishijima–Gell-Mann classification of elementary particles.

We wish to point out that this article only gives a summary of a longer paper which will be published later in the *Annals of the Institut Henri Poincaré*.

1. – New isotopic spin space.

As we said, previous works led us to take as a classical model of the internal structure of elementary particles the hyperspherical rotator ([2]) which can be described with an internal tetrad a_μ^ξ (μ, ξ, $\sim$ 1, 2, 3, 4) with another tetrad b_μ^ξ rotating around it. Tetrads are constituted by four unitary, orthogonal quadri-vectors, one, respectively $a_\mu^{(4)}$ and $b_\mu^{(4)}$ is a time-like vector, the three others being space-like vectors. Besides this rotator has the property that its inertial moment I is a constant.

As for the three-dimensional spherical rotator, whose movement is given more simply by the introduction of the Euler angles φ, θ, ψ, the descriptions of the b_μ^ξ rotations around a_μ^ξ will be made easier by the generalization of Euler parameters to space-time.

We have shown elsewhere ([1]), starting from the mathematical work of C. VAN WINTER ([5]) that the pseudo-angles:

$$(1) \quad \begin{cases} \omega^+ = \{\varphi_1 + i\varphi_2, \ \theta_1 + i\theta_2, \ \psi_1 + i\psi_2\} \\ \omega^- = \{\varphi_1 - i\varphi_2, \ \theta_1 - i\theta_2, \ \psi_1 - i\psi_2\} \end{cases} \qquad i = \sqrt{-1},$$

where φ_1, θ_1, ψ_1 are the ordinary space Euler angles, $i\varphi_2$, $i\theta_2$, $i\psi_2$ being the hyperbolic angles (varying from $-\infty$ to $+\infty$) and describing Lorentz transforms are really the generalization of Euler angles to space-time.

([4]) F. HALBWACHS, P. HILLION and J.-P. VIGIER: *Ann. Inst. Poincaré*, **16**, 115 (1959).

([5]) C. VAN WINTER: *Thésis* (Groningen, 1957).

Then we have:

$$(2) \qquad b_\mu^\xi = \Lambda_\eta^\xi(\omega^+, \omega^-) a_\mu^\eta \,,$$

where $\Lambda_\eta^\xi(\omega^+, \omega^-)$ are just the table of directing cosines of b_μ^ξ expressed in terms of this relativistic Euler angles.

Then, from a suggestion of EINSTEIN and MAYER ([6]), if we introduce self-dual bivectors:

$$(3) \qquad \begin{cases} B_k^{r\pm} = b_k^r b_4^4 - b_4^r b_k^4 \pm \varepsilon_{ijk} b_i^r b_j^4 \\[2mm] A_k^{r\pm} = a_k^r a_4^4 - a_k^4 a_4^r \pm \varepsilon_{ijk} a_i^r a_j^4 \end{cases} \qquad k, r \sim 1, 2, 3,$$

a transformation which brings a_μ^ξ over b_μ^ξ and defined by $\varphi_1, \theta_1, \psi_1, i\varphi_2, i\theta_2, i\psi_2$ also transforms A_k^{r+} into B_k^{r+} and A_k^{r-} into B_k^{r-} according to the expressions

$$(4) \qquad \begin{cases} B_k^{r+} = \Lambda_s^r(\omega^+) A_k^{s+} \,, & r, s \sim 1, 2, 3, \\[2mm] B_k^{r-} = \Lambda_s^r(\omega^-) A_k^{s-} \,, & k \sim 1, 2, 3, \end{cases}$$

with:

$$\omega^+ = \{\theta^+, \varphi^+, \psi^+\} \,, \qquad \omega^- = \{\theta^-, \varphi^-, \psi^-\} \,,$$

where $\Lambda_s^r(\omega)$ are just elements (but with complex angles) of the table of real three dimensional vectors b_k^r expressed in real Euler angles describing a three dimensional rotation:

$$b_k^r = \Lambda_s^r(\omega) a_k^s \,, \qquad r, s \sim 1, 2, 3;$$

$$\omega = \{\theta, \varphi, \psi\} \,, \qquad k \sim 1, 2, 3.$$

b_k^r and a_k^s being two trirectangle systems describing a spherical rotator.

From this we conclude that the relative motion of two tetrads a_μ^ξ and b_μ^ξ can be treated with the help or the one hand of A_k^{r+}, B_k^{r+} and ω^- and Λ_k^{r-}, B_k^{r-}, ω^- on the other hand.

Now if we introduce as Lagrangian describing the « internal » motion the Nakano term:

$$(5) \qquad L = -\tfrac{1}{4} I \dot{b}_\mu^\xi \dot{b}_\mu^\xi + \lambda_{\mu\nu}(b_\mu^\xi b_\nu^\xi - \delta_{\mu\nu}) \,,$$

we can express it in terms of B_k^{r+} and obtain

$$(6) \qquad L = -\tfrac{1}{4} I \dot{B}_k^{r+} \dot{B}_k^{r+} - \tfrac{1}{4} I \dot{B}_k^{r-} \dot{B}_k^{r-} + \lambda_{ij}^\pm (B_i^{r\pm} B_j^{\pm} - \delta_{ij}) \,,$$

which shows that the hyperspherical Lagrangian splits into the sum of two spherical complex conjugate rigid three dimensional rotators.

([6]) A. EINSTEIN and W. MAYER: *Sitz. Ber.*, 522 (Berlin, 1932).

If we now introduce the preceding relativistic Euler angles we get:

$$(7a) \qquad\qquad L = L^+ + L^-$$

with

$$(7b) \qquad\qquad L^\pm = I(\dot{\varphi}^{\pm^2} + \psi^{\pm^2} + \dot{\theta}^{\pm^2} + 2\,\dot{\varphi}^\pm\,\dot{\psi}^\pm \cos\theta^\pm) \;.$$

From this expression of L one deduces immediately the corresponding values of the momenta $P_{\omega^\pm} = \partial L/\partial\omega^\pm$ associated with the relativistic Euler angles $\omega^\pm$.

With the help of these values we calculate the relativistic Hamiltonian and obtain immediately:

$$(8) \qquad\qquad H = H^+ + H^- \;,$$

with

$$H^\pm = \frac{1}{2I}\left[P_{\theta^\pm}^2 + \frac{1}{\sin^2\theta^\pm}\,(p_{\varphi}^{2\pm} + p_{\psi}^{2\pm} + 2p_{\varphi^\pm}p_{\psi^\pm}\cos\theta^\pm) \right].$$

Naturally one also obtains (^{2}) the values of the projection of the rotator's angular momentum one the bivectors $A_k^{r\pm}$ and $B_k^{r\pm}$ which we denote by $S_k^\pm$ and $S_k'^\pm$. These projections are explicit functions of the $P_{\omega^\pm}$'s and $\omega^\pm$.

The quantization (^{3}) is performed as usual by substituting to $P_{\omega^\pm}$ the operators $-j\hbar(\partial/\partial\omega^\pm)$ with $j^2 = -1$.

The relativistic Hamiltonian which can be written:

$$(9) \qquad \left\{ \begin{aligned} H &= \frac{1}{2I}\,(S_k^+ S_k^+ + S_k^- S_k^-)\,, \\[2mm] &= \frac{1}{2I}\,(S_k'^+ S_k'^+ + S_k'^- S_k'^-)\,, \end{aligned} \right.$$

as well as $S_k^\pm$ and $S_k'^\pm$ become operators (*) and our problem is to study the wave fields associated to the Lie algebra of the complex rotation group.

We note immediately (^{5}) that the operators $J_k^\pm$ and $J_k'^\pm$ can be formally obtained from the infinitesimal operators of the real three dimensional rotators group by substituting our complex angles to the classical Euler angles.

These operators satisfy the structure relations of the infinitesimal group

$$(10a) \qquad\qquad [J_i^\pm,\; J_j^\pm] = -j\hbar\varepsilon_{ijk}J_k^\pm\,,$$

$$(10b) \qquad\qquad [J_i'^\pm,\; J_j'^\pm] = -j\hbar\varepsilon_{ijk}J_k'^\pm\,,$$

$$j = \sqrt{-1}\,, \qquad\qquad i,\,j,\,k \sim 1,\,2,\,3$$

(*) We call them henceforward $J_k^\pm$, $J_k'^\pm$.

and the commutation relations:

(11) $\qquad [J_i^+, J_j^-] = 0 , \quad [J_i'^+, J_i'^-] = 0 , \quad [J_i^\pm, J_j'^\pm] = 0 ,$

$$[J_k^+, J_k^+ J_3^+] = [J_k^+ J_k^+, J_3'^+] = 0 \quad \text{evidently} \quad [J_k'^+ J_k^-, J_3^+] = [J_k^- J_k^-, J_3'^+] = 0 ,$$

$$[J_k^- J_k^-, J_3^-] = [J_k^- J_k^-, J_3'^-] = 0 .$$

These last relations can be written:

(12) $\qquad [H, J_3^+] = [H, J_3^-] = [H, J_3'^+] = [H, J_3'^-] = 0$

so that J_3^+, J_3^-, $J_3'^+$, $J_3'^-$ are constant of the motion.

Now, to define functional space associated with these operators we look for eigenfunctions of six operators:

$$J_3^+ , \quad J_3'^+ , \quad (J^+)^2 , \quad J_3^- , \quad J_3'^- , \quad (J^-)^2 .$$

That is possible because of the preceding relations. We call these eigenfunctions:

$$U_{l^+, l^-}^{m^+, m^-, m'^+, m'^-}(\omega^+, \omega^-) ,$$

where l^+, l^- take integer or half-integer values independently and

$$m^+ , \quad m'^+ = -l^+ , \quad -l^+ + 1 , \quad ..., \quad l^+ - 1 , \quad l^+ ,$$

$$m^- , \quad m'^- = -l^- , \quad -l^- + 1 , \quad ..., \quad l^- - 1 , \quad l^- .$$

The functions $U_{l^+, l^-}^{m^+, m^-, m'^+, m'^-}(\omega^+, \omega^-)$ belong to $\mathscr{D}(l^+, l^-)$ representation of R_3^*. But as we have associated in our theory the parity to an operator in the automorphism group of R_3^*, it would be interesting to obtain vectorial spaces invariant under this operation. To reach this aim, let us introduce operators $S_k' = J_k'^+ + J_k'^-$ and $(S')^2 = S_k' S_k'$ and look for simultaneous eigenfunctions of:

$$J_3^+ , \quad (J^+)^2 , \quad J_3^- , \quad (J^-)^2 , \quad S_3' , \quad (S')^2 ,$$

with eigenvalues m^+, $l^+(l^+ + 1)$, m^-, $l^-(l^- + 1)$, m', $s'(s' + 1)$.

New eigenfunctions are obtained with the help of $U_{l^+, l^-}^{m^+, m^-, m'^+, m'^-}(\omega^+, \omega^-)$ and Clebsch-Gordan parameters:

(13) $\quad Z_{l^+, l^-, s'}^{m^+, m^-, m'}(\omega^+, \omega^-) = \sum_{-m'^+, -m'^-} (l^+, l^-, s', -m' | l^+, l^-, -m'^+, -m'^-) \cdot$

$$\cdot U_{l^+, l^-}^{m', m^-, m'^+, m'^-}(\omega^+, \omega^-) ,$$

where: $(l^+, l^-, s', -m' \,|\, l^+, l^-, -m'^+, -m'^-)$ are Clebsch-Gordan parameters with

$$s' = (l^+ + l^-), \quad (l^+ + l^- - 1), \quad \ldots, \quad |l^+ - l^-|,$$

$$m' = -s', \quad -s' + 1, \quad \ldots, \quad s' - 1, \quad s'.$$

For s' and m' fixed, we obtain basic vectors transforming under $\mathscr{D}(l^+, l^-)$ and $\mathscr{D}(l^-, l^+)$ representations of the complex rotation group. In the following we only use these basis vectors.

Each $Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$ is a solution of the equation

(14)
$$H\Psi = w\psi,$$

where

$$H = \frac{1}{2I}(J_k^+ J_k^+ + J_k^- J_k^-)$$

and

(15)
$$w = \frac{1}{I}[l^+(l^+ + s) + l^-(l^- + 1)]\hbar^2.$$

To conclude this paragraph, we point out that we have now to develop a new wave mechanics or internal wave mechanics characterized by its invariance under a three-dimensional complex rotation group (*). We will just mention the most important results, the complete theory being developed elsewhere.

2. – Internal quantum state vectors.

2·1. *Introductory remarks.* – Before giving results concerning this internal wave mechanics (the details and calculs are to be published elsewhere), let us recall certain properties of $Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$ functions. We used to define them $U^{m^+, m^-, m'^+, m'^-}_{l^+, l^-}(\omega^+, \omega^-)$ functions but ([5-7]):

(16)
$$U^{m^+, m^-, m'^+, m'^-}_{l^+, l^-}(\omega^+, \omega^-) = Y^{m^+, m'^+}_{l^+}(\omega^+) \, Y^{m^-, m'^-}_{l^-}(\omega^-),$$

with

$$Y^{m^+, m'^+}_{l^+}(\omega^+) = \Theta^{m^+, m'^+}_{l^+}(\theta^+) \exp\left[j(m^+\varphi^+ + m'^+\psi^+\right]$$

and

$$\Theta^{m^+, m'^+}_{l^+}(\theta^+) = \left(\sin\frac{\theta^+}{2}\right)^{-m^+ + m'^+} \left(\cos\frac{\theta^+}{2}\right)^{-m^+ - m'^+} \frac{d^{l^+ - m^+}}{d\left(\sin^2(\theta^+/2)\right)^{l^+ - m^+}} \cdot$$
$$\cdot \left(\sin^2\frac{\theta^+}{2}\right)^{l^+ - m'^+} \left(\cos^2\frac{\theta^+}{2}\right)^{l^+ + m'^+}.$$

Of course there is an analogous formula for $Y^{m^-, m'^-}_{l^-}(\omega^-)$.

(*) Whenever we speak hereafter of R_3^* group, we mean the group with the automorphism P.

([7]) P. HILLION and J.-P. VIGIER: *Ann. Inst. Poincaré*, **16**, 161 (1959).

Under the C.T.P. operations (*), $Z_{l^+,\ l^-,\ s'}^{m^+,\ m^-,\ m'}(\omega^+,\ \omega^-)$ transform according to the relations (^{8}):

(17)

$$P \quad Z_{l^+,\ l^-,\ s'}^{m^+,\ m^-,\ m'}(\omega^+,\ \omega^-) = (-1)^{l^+,\ l^-,\ s'}\ Z_{l^-,\ l^+,\ s'}^{m^+,\ m^-,\ m'}(\omega^+,\ \omega^-)\,,$$

$$T \quad Z_{l^+,\ l^-,\ s'}^{m^+,\ m^-,\ m'}(\omega^+,\ \omega^-) = (-1)^{d}\ \quad Z_{l^-,\ l^+,\ s'}^{-m^-,\ -m^+,\ -m'}(\omega^+,\ \omega^-)\,,$$

$$C \quad Z_{l^+,\ l^-,\ s'}^{m^+,\ m^-,\ m'}(\omega^+,\ \omega^-) = (-1)^{\alpha}\ \quad Z_{l^+,\ l^-,\ s'}^{-m^+,\ -m^-,\ -m'}(\omega^+,\ \omega^-)\,,$$

where

$$d = |m' - m^+ - m^-|$$

$$\alpha = l^+ + l^- + d - s'$$

so that, if we associate an internal state vector with s', m', values to a particle, then internal state vector with s', $-m'$, values have to be associated to the corresponding antiparticles.

According to our program, we are now in a position to introduce « internal state vectors ». Three conditions appear immediately.

a) Such vectors must evidently be built with $Z_{l^+,\ l^-,\ s'}^{m^+,\ m^-,\ m'}(\omega^+,\ \omega^-)$ functions belonging to the some irreducible representation according to the last remark of the preceding paragraph. Let us denote these internal state vectors by Φ.

b) We immediately see that the equation:

$$H\Phi = W\Phi\,,$$

where H is given in (14), is separable, so that Φ satisfies simultaneously the two equations:

(18)

$$(J_k^+ J_k^+ - \chi^{+2})\Phi = 0\,,$$

$$(J_k^- J_k^- - \chi^{-2})\Phi = 0\,,$$

with

$$\chi^\pm = l^\pm(l^\pm + 1)\hbar^2\,.$$

c) « Internal state vectors » have a tensorial or spinorial character according to the integer or half-integer value of the sum $l^+ + l^-$, these two quantum numbers giving the dimensions of the irreducible representation.

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(*) We recall that here just as P is an automorphism of P_3^*, C and T are involutions in the automorphism group. The C.T.P. theorem expresses that a product of two involutions is an automorphism.

(^{8}) P. HILLION and J.-P. VIGIER: *Ann. Inst. Poincaré*, **16**, 217 (1959).

To obtain « internal state vectors » we shall find a fortunate help in the following circumstances:

a) First we limit ourselves to the lower grade representations:
For $l^+ + l^-$ integer, to $\mathscr{D}(\frac{1}{2}, \frac{1}{2})$, $\mathscr{D}(1, 0)$ and $\mathscr{D}(0, 1)$.
For $l^+ + l^-$ half-integer to $\mathscr{D}(\frac{1}{2}, 0)$, $\mathscr{D}(0, \frac{1}{2})$, $\mathscr{D}(\frac{1}{2}, 1)$ and $\mathscr{D}(1, \frac{1}{2})$.

b) Secondly, the isomorphism between Lorentz group and complex rotation group ([9]), allows to forecast the form of these « internal state vectors ». For instance we get four-vectors for $\mathscr{D}(\frac{1}{2}, \frac{1}{2})$, and self-dual antisymmetric tensors for

$$\mathscr{D}(1, 0) \text{ and } \mathscr{D}(0, 1), \quad ..., \quad \text{etc.}$$

c) Besides, the dimensions d_4 of the representation matrix $\mathscr{D}^{(l^+, \, l^-)}(\omega^+, \, \omega^-)$ are well-known ([10]) and internal state vectors are, for basic vectors, the elements of any row in this matrix so that with each row a vectorial space is constructed with dimensions d_4 in which are defined d_4 independent vectors.

We recall the following results:

representation	d_4
$\mathscr{D}(\frac{1}{2}, \frac{1}{2})$	4
$\mathscr{D}(1, 0)$ and $\mathscr{D}(0, 1)$	3
$\mathscr{D}(\frac{1}{2}, 0)$ and $\mathscr{D}(0, \frac{1}{2})$	2
$\mathscr{D}(\frac{1}{2}, 1)$ and $\mathscr{D}(1, \frac{1}{2})$	6

As each row of the matrix $\mathscr{D}^{l^+, \, l^-}(\omega^+, \omega^-)$ is fixed with help of two numbers s', m', we have just to determine their values with

$$s' = (l^+ + l^-), \quad (l^+ + l^- - 1), \quad ..., \quad |l^+ - l^-|,$$

$$m' = -s', \quad -s' + 1, \quad ..., \quad s' - 1, \quad s'.$$

In the following we only consider the internal state vectors corresponding to the lesser value $s' = |l^+ - l^-|$. This restriction can be easily justified by considering the spin but we shall not study this question here because it is then necessary to introduce Lagrangian formalism and that is now beyond our program. We shall prove elsewhere that this hypothesis is consistent with the usual spin of elementary particles. Let us note that for $\mathscr{D}(1, 0)$ and $\mathscr{D}(0, 1)$ representations, s' takes the only value 1, so that m' has three values $1, -1, 0$. We also keep $m' = 0$ for spin considerations.

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[9] E. CARTAN: *Leçon sur la théorie des spineurs* (Paris).
[10] See for instance E. M. CORSON: *Introduction to tensors, spinors and relativistic wave equations* (New York).

2'2. *Representations with* $l^+ + l^-$ *integer.* – Let us begin with $\mathscr{D}(\tfrac{1}{2}, \tfrac{1}{2})$ representation and $s' = 0$.

After the preceding considerations we can associate four states corresponding to four independent fourvectors.

These vectors will be denoted $A_\mu^{(1)}$, $A_\mu^{(2)}$, $A_\mu^{(3)}$ and $A_\mu^{(4)}$. Their exact form can be found in the enclosed table of internal state vectors.

Each of these vectors satisfy the system:

$$(20) \quad \begin{cases} (J_k^+ J_k^+ - \chi^2) A_\mu^\xi = 0 \,, & \mu, \; \xi \sim 1, \, 2, \, 3, \, 4, \\ (J_k^- J_k^- - \chi^2) A_\mu^\xi = 0 \,, & \chi^2 = \tfrac{3}{4}\hbar^2. \end{cases}$$

Let us consider now the $\mathscr{D}(1, 0)$ and $\mathscr{D}(0, 1)$ representations, with $m' = 0$. As we noted before, here internal state vectors are antisymmetric self-dual tensors:

$$F_{ij}^+ \quad \text{associated with} \quad \mathscr{D}(1, 0)\,,$$

$$F_{ij}^- \quad \text{associated with} \quad \mathscr{D}(0, 1)\,.$$

We can replace these tensors by pseudo-vectors A_k^+ and A_k^-.

$$A_k^+ = F_{ij}^+\,, \qquad A_k^- = F_{ij}^-\,,$$

where i, j, k, is a circular permutation of 1, 2, 3.

There are three A_k^+ vectors and three A_k^- vectors; we label them A_k^{r+} and A_k^{r-} $r \sim 1, \, 2, \, 3$.

They respectively satisfy the equations:

$$(21) \quad \begin{cases} (J_k^+ J_k^+ - \chi^2) A_k^{r+} = 0 \,, \\ (J_k^- J_k^- - \chi^2) A_k^{r-} = 0 \,, \end{cases} \qquad \chi^2 = 2\hbar^2.$$

The exact form of $A_k^{(r)+}$ is given in the table. One passes from $A_k^{(r)+}$ to $A^{(r)-}$ by application of the operation P defined by (17).

2'3. *Representations with* $l^+ + l^-$ *half-integer.* – With the $\mathscr{D}(\tfrac{1}{2}, 0)$ and $\mathscr{D}(0, \tfrac{1}{2})$ representations, s' takes the unique value $\tfrac{1}{2}$ (so that $m' = \pm \tfrac{1}{2}$), we can associate four two-component spinors to $\mathscr{D}(\tfrac{1}{2}, 0)$ and four other two-component spinors to $\mathscr{D}(0, \tfrac{1}{2})$, which we shall consider as our internal state vectors, that is:

representation $\mathscr{D}(\tfrac{1}{2}, 0)$	$m' = \tfrac{1}{2}$	φ^s and	φ_r
	$m' = -\tfrac{1}{2}$	ψ^s and	ψ_r
representation $\mathscr{D}(0, \tfrac{1}{2})$	$m' = \tfrac{1}{2}$	$\varphi_{\dot r}$ and	$\varphi^{\dot s}$
	$m' = -\tfrac{1}{2}$	$\psi_{\dot r}$ and	$\psi^{\dot s}$

Here r, s, $\dot r$, $\dot s \sim 1, \, 2$.

Representation	$s' = $			

$D(\tfrac{1}{2} \cdot \tfrac{1}{2})$

$$A_2^{(1)} = -\frac{1}{2i}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_3^{(1)} = \frac{1}{2}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} + Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_4^{(1)} = \frac{1}{2i}(Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0})$$

$$A_2^{(2)} = \frac{1}{2i}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} + Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_3^{(2)} = -\frac{1}{2}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} + Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_4^{(2)} = \frac{1}{2i}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$

$$A_2^{(3)} = \frac{1}{2i}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_3^{(3)} = -\frac{1}{2}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_4^{(3)} = -\frac{1}{2i}(Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0} + Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0})$$

$$A_2^{(4)} = -\frac{1}{2}(Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0})$$
$$A_3^{(4)} = -\frac{1}{2}(Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0} - Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0})$$
$$A_4^{(4)} = \frac{1}{2}(Z_{\frac{1}{2}\frac{1}{2}0}^{1-\frac{1}{2}0} + Z_{\frac{1}{2}\frac{1}{2}0}^{1\frac{1}{2}0})$$

$D(1,0)$

$$A_1^{(1),+} = \frac{i}{2}(Z_{101}^{100} - Z^{-100})$$
$$A_2^{+(1)} = \frac{1}{2}(Z_{101}^{100} + Z^{-100})$$
$$A_3^{(1)+} = 0$$

$$A_1^{(2)+} = \frac{i}{2}(Z_{101}^{100} - Z^{-100})$$
$$A_2^{(2)+} = \frac{1}{2}(Z_{101}^{100} + Z^{-100})$$
$$A_3^{(2)+} = 0$$

$$A_1^{(3)+} = 0$$
$$A_2^{(3)+} = 0$$
$$A_3^{(3)+} = Z_{101}^{000}$$

$D(0,\tfrac{1}{2})$

$$\varphi^s = \begin{pmatrix} Z_{0\frac{1}{2}\frac{1}{2}}^{0\frac{1}{2}\frac{1}{2}} \\ Z_{0\frac{1}{2}\frac{1}{2}}^{0-\frac{1}{2}\frac{1}{2}} \end{pmatrix} m' = -\frac{1}{2}
\qquad
\varphi_r = \begin{pmatrix} -Z_{0\frac{1}{2}\frac{1}{2}}^{0\frac{1}{2}-\frac{1}{2}} \\ Z_{0\frac{1}{2}\frac{1}{2}}^{0\frac{1}{2}\frac{1}{2}} \end{pmatrix}$$

$$\varphi^3 = \begin{pmatrix} -Z_{0\frac{1}{2}\frac{1}{2}}^{0-\frac{1}{2}-\frac{1}{2}} \\ -Z_{0\frac{1}{2}\frac{1}{2}}^{0-\frac{1}{2}\frac{1}{2}} \end{pmatrix} m' = -\frac{1}{2}
\qquad
\psi_i = \begin{pmatrix} Z_{0\frac{1}{2}\frac{1}{2}}^{0-\frac{1}{2}-\frac{1}{2}} \\ -Z_{0\frac{1}{2}\frac{1}{2}}^{0\frac{1}{2}-\frac{1}{2}} \end{pmatrix}$$

Representation $D(\tfrac{3}{2}, 1)$, $m' = \tfrac{3}{2}$

$$A_1^{1\frac{1}{2}} = \frac{1}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1\frac{1}{2}} + Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1\frac{1}{2}})$$
$$A_1^{2\frac{1}{2}} = \frac{1}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1\frac{1}{2}} + Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1\frac{1}{2}})$$
$$A_1^{3\frac{1}{2}} = 0$$

$$A_2^{1\frac{1}{2}} = \frac{i}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1\frac{1}{2}} - Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1\frac{1}{2}})$$
$$A_2^{2\frac{1}{2}} = \frac{i}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1\frac{1}{2}} + Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1\frac{1}{2}})$$
$$A_2^{3\frac{1}{2}} = 0$$

$$A_3^{1\frac{1}{2}} = 0$$
$$A_3^{2\frac{1}{2}} = 0$$
$$A_3^{3\frac{1}{2}} = \begin{pmatrix} Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}0\frac{1}{2}} \\ Z_{\frac{3}{2}1\frac{1}{2}}^{-\frac{1}{2}0\frac{1}{2}} \end{pmatrix}$$

$$A_1^{1-\frac{1}{2}} = \frac{1}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1-1\frac{1}{2}} + Z_{\frac{1}{2}1\frac{1}{2}}^{\frac{1}{2}-1-1\frac{1}{2}})$$
$$A_1^{2-\frac{1}{2}} = \frac{1}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1-1\frac{1}{2}} + Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1-1\frac{1}{2}})$$
$$A_1^{3-\frac{1}{2}} = 0$$

$$A_2^{1-\frac{1}{2}} = \frac{i}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1-1\frac{1}{2}} - Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1-1\frac{1}{2}})$$
$$A_2^{2-\frac{1}{2}} = \frac{i}{2}(Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}1-1\frac{1}{2}} + Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}-1-1\frac{1}{2}})$$
$$A_2^{3-\frac{1}{2}} = 0$$

$$A_3^{1-\frac{1}{2}} = 0$$
$$A_3^{2-\frac{1}{2}} = 0$$
$$A_3^{3-\frac{1}{2}} = \begin{pmatrix} Z_{\frac{3}{2}1\frac{1}{2}}^{\frac{1}{2}0\frac{1}{2}} \\ Z_{\frac{3}{2}1\frac{1}{2}}^{-\frac{1}{2}0\frac{1}{2}} \end{pmatrix}$$

The exact form of φ^s, φ_r, ψ^s, ψ_r is given in the table. One passes from $\mathscr{D}(0, \frac{1}{2})$ to $\mathscr{D}(\frac{1}{2}, 0)$ by application of the P operation and permutation of the spinor components.

These state vectors satisfy the relations:

$$(26) \qquad (\sigma_k J_k^+ - \chi_1)\varphi^{\dot{s}} = 0 , \qquad (\sigma_k J_k^+ - \chi_2)\varphi_{\dot{r}} = 0 .$$

Same thing for $\psi^{\dot{s}}$ and $\psi_{\dot{r}}$.

$$(27) \qquad (\sigma_k J_{k-} - \chi_1)\varphi^s = 0 , \qquad (\sigma_k J_{k-} - \chi_2)\varphi_r = 0 .$$

We get the same relations for ψ^s, and ψ_r where

$$\chi_1 = \frac{\hbar}{2} + \sqrt{\chi^2 + \frac{\hbar^2}{4}} , \qquad \chi_2 = \frac{\hbar}{2} - \sqrt{\chi^2 + \frac{\hbar^2}{4}} , \qquad \chi^2 = \frac{3}{4}\hbar^2 .$$

σ_k being the Pauli 2×2 matrices.

Now we come to the $\mathscr{D}(\frac{1}{2}, 1)$ and $\mathscr{D}(1, \frac{1}{2})$ representations, with $s' = \frac{1}{2}$. We can associate twelve independent internal state vectors with each of the two representations, six for each value of m'.

It is almost evident that these state vectors must combine the properties of the self-dual antisymmetric tensors of the $\mathscr{D}(0, 1)$ representation with those of the two-component spinors associated with the $\mathscr{D}(\frac{1}{2}, 0)$ representation. Besides we can prove that they satisfy the simultaneous system:

$$(28a) \qquad \text{For} \quad \mathscr{D}(\frac{1}{2}, 1): \quad \begin{cases} (\sigma_k J_k^+ - \chi_1)\Phi = 0 , \\ (J_k^- J_k^- - \chi_2)\Phi = 0 . \end{cases}$$

$$(28b) \qquad \text{For} \quad \mathscr{D}(1, \frac{1}{2}): \quad \begin{cases} (J_k^+ J_k^+ - \chi^2)\Phi = 0 , \\ (\sigma_k J_k^- - \chi_1)\Phi = 0 , \end{cases}$$

$$\chi_1 = \frac{\hbar}{2} \pm \sqrt{\chi'^2 + \frac{\hbar^2}{4}} , \qquad \chi'^2 = \frac{3}{4}\hbar^2 , \qquad \chi^2 = 2\hbar^2 .$$

But as it is well-known, to the $\mathscr{D}(\frac{1}{2}, 1)$ and $\mathscr{D}(1, \frac{1}{2})$ representations correspond third rank spinors which write in Van der Waerden's terminology:

$$\Phi_{rs}\dot{i} \qquad \text{for} \quad \mathscr{D}(1, \frac{1}{2}) ,$$
$$\Phi^{\dot{r}\dot{s}}t \qquad \text{for} \quad \mathscr{D}(\frac{1}{2}, 1) .$$

Now the problem is to find internal state vectors constructed with these third rank spinors and satisfying the preceding equations. In the Rarita-

Schwinger formalism ([11]) one defines the quantities:

$$\Phi^r_{\ \mu} = \tfrac{1}{2}\sigma_{\mu t}^{\ \ \dot{u}}\Phi^{rt}_{\ \ \dot{u}}\,,$$

$$\Phi_{\dot{s},\mu} = \tfrac{1}{2}\sigma_{\mu u}^{\ \ i}\Phi_{\dot{s}t}^{..\ u}\,.$$

But it is not difficult to see that they are not solutions of equations (28). We are thus led to use the following state vectors:

$$A^r_{ij} = \tfrac{1}{2}\sigma_{iu}^{\ \ \dot{s}}\sigma_j^{\ ut}\Phi_{\dot{s}t}^{..\ r}\,,\qquad B_{\dot{s},\,ij} = \tfrac{1}{2}\sigma_{\dot{i}\,r}^{\ \ \dot{u}}\sigma_{j\dot{u}t}\Phi^{rt}_{\ \ \dot{s}}\,,$$

that is if i, j, k is a circular permutation of 1, 2, 3,

(29)
$$
\left\{
\begin{aligned}
A^r_{\ k} &= \tfrac{1}{2}\sigma_{iu}^{\ \ \dot{s}}\sigma_j^{\ ut}\begin{pmatrix}\Phi_{\dot{s}t}^{..\,1}\\[4pt]\Phi_{\dot{s}t}^{..\,2}\end{pmatrix}\\[12pt]
B_{\dot{s},\,k} &= \tfrac{1}{2}\sigma_{\dot{i}\,r}^{\ \ \dot{u}}\sigma_{j\dot{u}t}\begin{pmatrix}\Phi^{1}_{\ \dot{s}t}\\[4pt]\Phi^{2}_{\ \dot{s}t}\end{pmatrix}
\end{aligned}
\right.
$$

So that we get;

where we label internal state vectors A^{xy}_k and B^{xy}_k with

$$k,\,x \sim 1,\,2,\,3\,,\qquad y = \pm\tfrac{1}{2}\,.$$

a) For the representation $\mathscr{D}(\tfrac{1}{2},1)$ with $m' = \tfrac{1}{2}$.

We find six internal state vectors which we shall denote $A^{1\frac{1}{2}}_k$, $A^{2\frac{1}{2}}_k$, $A^{3\frac{1}{2}}_k$, $A^{1-\frac{1}{2}}_k$, $A^{2-\frac{1}{2}}_k$, $A^{3-\frac{1}{2}}_k$ with components formed by a production of self-dual skew tensor components and two component spinors. They are expressed in the table.

As before we obtain the corresponding state vectors $B^{r\frac{1}{2}}_k$, $B^{r-\frac{1}{2}}_k$ from $A^{r\frac{1}{2}}_k$, $A^{r-\frac{1}{2}}_k$ by application of the P operation and permutation of the spinor components.

To conclude this paragraph let us remark that one could generalize the preceding procedure to any $\mathscr{D}(l^+, l^-)$ and $\mathscr{D}(l^-, l^+)$ representations but our model evidently suggests that if our prospective identification of elementary particle structure with internal state vector is valid one must expect to find them in the « lowest internal states » developped in this section. Indeed, as we shall see in the next section the preceding representations furnish all known elementary particles. As it will be seen later, only the photon does not fit in this theory.

([11]) See for instance H. UMEZAWA: *Quantum Field Theory* (Amsterdam).

3. – Elementary particle classification.

According to our program, we shall now study how the internal state vectors behave in the transformation defined by the operators J_3^+, J_3^-, S_3' which have been shown to be constants of motion. Applied to the functions $Z_{l+,\ l-,\ s'}^{m+,\ m-,\ m'}(\omega^+, \omega^-)$, these operators lead to the following results:

(34)
$$
\begin{cases}
J_3^+ \, Z_{l+,\ l-,\ s'}^{m+,\ m-,\ m'}(\omega^+, \omega^-) = m^+ \, Z_{l+,\ l-,\ s'}^{m+,\ m-,\ m'}(\omega^+, \omega^-)\,, \\[2mm]
J_3^- \, Z_{l+,\ l-,\ s'}^{m+,\ m-,\ m'}(\omega^+, \omega^-) = m^- \, Z_{l+,\ l-,\ s'}^{m+,\ m-,\ m'}(\omega^+, \omega^-)\,, \\[2mm]
S_3' \, Z_{l+,\ l-,\ s'}^{m+,\ m-,\ n'}(\omega^+, \omega^-) = m' \, Z(\omega^+, \omega^-)\,,
\end{cases}
$$

in a system of units where $\hbar$ is taken equal to 1.

In the vectorial space of the internal wave functions we shall have to consider the operations $I_3^+ J_3^+$, $I_3^- J_3^-$, S_3', where I_3^+ and I_3^- are the matrices of infinitesimal rotation about the third axis.

We shall show that:

1) these three operators respectively correspond to the isotopic spin, the strangeness and the Fermionic number (or, more exactly, for the two last ones, to one half of these quantities);

2) their eigenvalues make it possible to establish a classification of the internal state vectors reproducing all the essential characteristics of the Nishijima–Gell-Mann scheme.

Let us first write the table of the infinitesimal operators I_3^+ and I_3^- associated to each representation:

(35)   a) $\mathscr{D}(\tfrac{1}{2}, \tfrac{1}{2})$:
$$
I_3^+ = \begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}, \qquad
I_3^- = \begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}.
$$

(36a)   b) $\mathscr{D}(1, 0)$:
$$
I_3^+ = \begin{vmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},
$$

$$(36b) \quad c) \quad \mathscr{D}(0,1): \qquad I_3^- = \begin{vmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

$$(37) \quad d) \quad \mathscr{D}(\tfrac{1}{2},0) \qquad I_3^+ = \sigma_3, \qquad\qquad \mathscr{D}(0,\tfrac{1}{2}), \qquad\qquad I_3^- = \sigma_3.$$

$$e) \quad \mathscr{D}(1,\tfrac{1}{2}): \qquad I_3^+ = \begin{vmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad\qquad I_3^- = \sigma_3.$$

$$(38) \quad f) \quad \mathscr{D}(\tfrac{1}{2}\,1): \qquad I_3^+ = \sigma_3, \qquad I_3^- = \begin{vmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Now let us present the results in the following way:

In the first column, we shall write the internal state vectors as they were given in the preceding section; in the three other columns, the eigenvalues of the operators $I_3^+ J_3^+$, $I_3^- J_3^-$, S_3' of the corresponding internal state vectors; in the fifth column, the value Q of the sum of the last three numbers. Finally, in the last column, we enter the particle, characterized, according to the Nishijima–Gell-Mann classification ([1]) by the four members in columns 2, 3, 4, 5.

The computations leading to the table shown underneath are immediate and may be readily verified:

a) *Representation* $\mathscr{D}(\tfrac{1}{2},\tfrac{1}{2})$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
$A_\mu^{(1)}$	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	0	-1	K^-
$A_\mu^{(2)}$	$\tfrac{1}{2}$	$\tfrac{1}{2}$	0	1	K^+
$A_\mu^{(3)}$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	0	0	$\widetilde{K}^0$
$A_\mu^{(4)}$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	0	0	K^0

It is exactly the Nishijima–Gell-Mann scheme for the K-mesons; we can very well see that K^+ is the anti-particle of K^- and that $\widetilde{K}^0$ is the anti K^0.

Furthermore, we can check that

$$I_3^+ J_3^+ = I_{30p}\,, \qquad I_{30p} \quad \text{being the isotopic spin}$$

$$I_3^- J_3^- = 2S_{0p}\,, \qquad S_{0p} \quad \text{being the strangeness}$$

$$S_3' = 2N_{0p}\,, \qquad N_{0p} \quad \text{being the baryonic number operator}$$

This may be also verified for the following tables.

b) *Representation* $\mathscr{D}(1, 0)$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
$A_k^{(1)}$	1	0	0	1	π^+
$A_k^{(2)}$	-1	0	0	-1	π^-
$A_k^{(3)}$	0	0	0	0	π^0

c) *Representation* $\mathscr{D}(0, 1)$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
$B_k^{(1)}$	0	1	0	1	—
$B_k^{(2)}$	0	-1	0	-1	—
$B_k^{(3)}$	0	0	0	0	—

In the Nishijima–Gell-Mann classification, no particle is associated with $\mathscr{D}(0, 1)$. However, new particles have been recently observed and some authors [12] consider the possibility of their association with $\mathscr{D}(0, 1)$.

d) *Representation* $\mathscr{D}(\frac{1}{2}, 0)$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
$\psi_r^\cdot$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	e
$\psi_s^\cdot$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	ν

[12] J. C. POLKINGHORNE and A. SALAM: *Nuovo Cimento,* **15**, 160 (1960).

It is to be recalled that the Nishijima–Gell-Mann scheme gives no information concerning the leptons so that it is not possible here to interpret $I_3^+ J_3^+$ and $I_3^- J_3^-$ as the isotopic spin and strangeness. Replacing vectors ψ^s and ψ_r by φ^s and φ_r would yield the antiparticles; the consequence in the table above is a change of the sign of all numbers.

e) *Representation* $\mathcal{D}(0, \tfrac{1}{2})$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
φ^r	0	$\tfrac{1}{2}$	$\tfrac{1}{2}$	1	μ^+
φ_s	0	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	0	ν_μ (*)

A consequence of the preceding table is that μ^- should be considered as the particle and μ^- as the anti-particle.

f) *Representation* $\mathcal{D}(\tfrac{1}{2}, 1)$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
$A_k^{1,\frac{1}{2}}$	$\tfrac{1}{2}$	1	$\tfrac{1}{2}$	2	
$A_k^{1,-\frac{1}{2}}$	$-\tfrac{1}{2}$	1	$\tfrac{1}{2}$	1	
$A_k^{2,\frac{1}{2}}$	$\tfrac{1}{2}$	-1	$\tfrac{1}{2}$	0	Ξ^0
$A_k^{2,-\frac{1}{2}}$	$-\tfrac{1}{2}$	-1	$\tfrac{1}{2}$	-1	Ξ^-
$A_k^{3,\frac{1}{2}}$	$\tfrac{1}{2}$	0	$\tfrac{1}{2}$	1	p
$A_k^{3,-\frac{1}{2}}$	$-\tfrac{1}{2}$	0	$\tfrac{1}{2}$	0	n

We find three isotopic spin doublets of which two only have a correspondence in the Nishijima–Gell-Mann classification. But it will be noticed that

(*) *Note added in proof.* - In a first version of this paper we had placed Λ^0 along with μ^- in this Table. However after discussion with Prof. DALLAPORTA, DE BROGLIE, PONTECORVO and NAKANO its seems preferable to use this possibility to introduce into the lepton scheme the second neutrino (which looks necessary to interpret weak interactions) proposed by PONTECORVO (*Journ. Exp. Theor. Phys.*, **37**, 1751 (1959)). As a consequence our Tables do not contain Λ^0 any more. Λ^0 should then be considered as an unelementary particle (in Sakata's sense) or associated with another representation. $D(0, e^-)$ such as $D(0, \tfrac{3}{2})$ which gives isotopic singlets.

the one not associated with a particle has one of its members with a charge 2. Then, if it is assumed that the charge can have but the values 1, 0, or -1, and that if one of the terms of the doublet is physically impossible, the other one is also impossible, we are not led to a greater number of particles. The anti-particles correspond to the other internal state vectors (with $m' = -\frac{1}{2}$) of the $\mathscr{D}(\frac{1}{2}, 1)$ representation.

g) *Representation* $\mathscr{D}(1, \frac{1}{2})$:

State vector	$I_3^+ J_3^+$	$I_3^- J_3^-$	S_3'	Q	Particle
$B_k^{1,\frac{1}{2}}$	1	$\frac{1}{2}$	$\frac{1}{2}$	2	
$B_k^{2,\frac{1}{2}}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	
$B_k^{3,\frac{1}{2}}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	
$B_k^{1,-\frac{1}{2}}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	1	Σ^+
$B_k^{2,-\frac{1}{2}}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	-1	Σ^-
$B_k^{3,-\frac{1}{2}}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	Σ^0

Here we have two isotopic spin triplets of which one alone has a correspondence in the Nishijima–Gell-Mann scheme. The other one may be eliminated by the same reasoning as was applied to the alternate doublet of $\mathscr{D}(\frac{1}{2}, 1)$.

Now, before we conclude, we must answer two questions:

1) Concerning the charge Q. In this paper we did not say anything about the Lagrangian formalism associated with these internal waves (the corresponding study will be published separately). But if the gauge invariance of the first kind is imposed, according to the well-known Pauli hypothesis, to the internal state vector Φ, we are led to the relation:

$$(39) \qquad \delta\Phi = i(I_3^+ J_3^+ + I_3^- J_3^- + S_3')\,\Phi$$

and the charge operators comes out to be:

$$(40) \qquad Q_{op} = I_3^+ J_3^+ + I_3^- J_3^- + S_3',$$

which justifies the Nishijima–Gell-Mann formula.

2) About the spin of these wave vectors. It will be shown later that, from the Lagrangian formalism, can be deduced, in the case of the fermions,

a tensor analogous to the Belinfante-Rosenfeld tensor, leading to the spin pseudo-vector:

$$S_k = \hbar \widetilde{\Phi} \sigma_h \Phi$$

so that the spin assumes the values $\pm h/2$ for the wave vectors considered.

In the case of the bosons, it is easy to see that, for $\mathscr{D}(\frac{1}{2}, \frac{1}{2})$, $\mathscr{D}(1, 0)$ and $\mathscr{D}(0, 1)$, m' being equal to zero, the wave vectors have only one independent component in the rest-system ($\omega^+ = \omega^+ = \omega$). This corresponds to zero spin.

Conclusion.

It is clear that, if our basic starting point and subsequent calculations correspond to physical reality, many new problems and lines of research are opened which must be investigated.

The first line is evidently connected with the physical interpretation and discussion of the preceding results. As we know ([2]) our basic extended particle model constitutes a realistic model of Yukawa's bilocal model and is closely related with the new conception of a possible subquantum level of matter introduced by BOHM and one of us (J.-P. V.) ([13]). Researches are under way in Bristol and the Institute Henri Poincaré in order to understand the meaning of quantization in terms of real physical properties of that new level.

The second line of research results directly from our calculations. Since we know the exact structure and have symmetry properties of every state vector it is evidently easier to study different types of internal interaction relativistic Hamiltonians corresponding to different types of couplings. In our model the conservation or variation of i_3 and S are connected with conservation or variation of internal angular moments so that we now dispose of a concrete model to study different types of possible interactions.

The last line is related with quantization of the internal motion of relativistic rotators. As we have seen our basic starting point is to assume that all types of elementary particles correspond to internal quantized fields, represented within a certain approximation by hyperspherical relativistic rotators. So our problem can be expressed in the following form: what are the external quantized motions of relativistic rotators with internal motion characterized by a certain quantum state vector calculated in this paper?

The answer is facilitated by the remark that the « internal group » of three-dimensional complex rotations and its representations are isomorphic to the

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([13]) D. BOHM and J.-P. VIGIER: *Phys. Rev.*, **96**, 208 (1956).

Lorentz group and its irreducible representations: preliminary research then indicates that internal and external state functions should belong to isomorphic representations. This would justify a recent suggestion made with DE BRO-GLIE ([14]) that relativistic rotators and not point particles are the correct classical starting point for quantum theory. The consequences would evidently be far reaching since the model would thus provide simultaneously not only the external wave field but also the internal structure and internal quantum numbers characterizing any given elementary particle. These considerations will be developed in a subsequent paper.

* * *

([14]) L. DE BROGLIE, P. HILLION and J.-P. VIGIER: *Compt. Rend.*, **249**, 2255 (1959).

RIASSUNTO (*)

Partiamo dalla nuova idea che lo spazio dello spin isotopico è invariante rispetto al gruppo delle rotazioni coniugate complesse tridimensionali e giustifichiamo questa ipotesi sulla base di un modello classico di particelle estese descritte da rotatori relativistici. Poi esprimiamo in termini dei parametri di Eulero (generalizzazione relativistica degli angoli reali tridimensionali di Eulero) gli operatori infinitesimali dell'algebra di Lie associati a questo gruppo ed otteniamo, nello spazio generale delle autofunzioni corrispondenti, sottospazi dimensionali finiti associati a rappresentazioni irriducibili del gruppo completo di rotazione complessa. In ogni sottospazio si definisce facilmente una serie di « vettori di stato isotopici o interni » corrispondenti a differenti specie di particelle elementari in modo che ogni famiglia di particelle sia connessa ad una rappresentazione irriducibile di questo gruppo. In questo schema i numeri di stranezza isotopica e fermionici corrispondono a certi operatori infinitesimali di rotazione e si giustifica (con nuovi aspetti qualitativi) la classificazione empirica di Nishijima–Gell-Mann delle particelle elementari.

Magnetic Viscosity and Annealing in Magnetic Field [*].

G. BIORCI, A. FERRO [**] and G. MONTALENTI

Centro Studi Elettrofisica, Istituto Elettrotecnico Nazionale Galileo Ferraris - Torino

(ricevuto il 9 Maggio 1960)

Summary. — In the present work we show that a sufficient, but not necessary condition, such as a magnetic material exibits magnetic annealing is that it presents also, in a proper temperature range, diffusion magnetic viscosity. Besides, if H_t is the maximum value of the viscosity field, the energy $H_t J_s$ is nearly equal to the anisotropy energy K_u induced by annealing in a magnetic field, at least for measurements performed at the same temperature. It is also shown that the anomalies of specific heat, associated to the process of annealing, are so small that they cannot be observed experimentally. The experimental verifications are made on iron containing interstitial carbon. The values of the energy $H_t J_s$, obtained from the measure of the maximum value of the viscosity field, are lower than, but of the same order of magnitude as those of the induced anisotropy constant, measured directly. If one considers the poor accuracy with which K_u could be determined one can consider the agreement obtained to be satisfactory.

1. – Introduction.

Often it happens that the magnetization curve, drawn immediately after demagnetization, does not coincide with the one drawn after a certain time interval. If one observes this phenomenon, one must say, after NÉEL [1], that

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[1] L. NÉEL: *Journ. Phys. et Rad.*, **13**, 250 (1952).

there exists diffusion viscosity. Phenomena of this type have been observed many years ago. SNOEK [2] has experimentally demonstrated that in iron they are due to the presence of interstitial atoms of carbon and nitrogen. NÉEL [2] has given a brilliant theoretical interpretation of the effect. The maximum horizontal displacement H_i between the two curves is defined as the maximum value of the viscosity field. This quantity is directly related, according to NÉEL [1], to the quantity of interstitial carbon or nitrogen existent in the iron specimen.

The authors have theoretically demonstrated and experimentally verified that phenomena of diffusion viscosity at high temperature can also be due to a relaxation at the grain boundaries [3], to the motion of the dislocations [5], to rotation of couples of atoms in alloys [4] (e.g. in Fe–Si to rotation of couples Si–Si). The maximum value of the viscosity field in the first two cases is of the order of 1 A/m, in the alloys it can easily reach some ten A/m.

The possibility of improving the value of the maximum permeability and of squaring the loop by annealing in a magnetic field in the alloys is known by long [6]. Attemps of interpretation of the phenomenon have been given by BOZORTH [7 8] (1934); CHIKAZUMI [9] (1950); KAYA [10] (1953); NÉEL [11] (1953-54); TANIGUCHI [12-14] (1954-55). The last authors try to interpret the annealing in a magnetic field as due to phenomena of short-range order appearing in the alloy. It is obvious that here are not considered the cases in which we have, during the annealing in a magnetic field, formation of anisotropic precipitates (e.g. Alnico) aligned along the field axis.

BOZORTH [15] suggests that interpretations of this kind are contradictory to experimental facts, for one does not observe the characteristic anomalies of specific heath appearing in the usual order-disorder transformations.

[2] J. L. SNOEK: *Physica*, **5**, 663 (1938).

[3] G. BIORCI, A. FERRO and G. MONTALENTI: *Solid State Physics*. Editors H. DÉSIRANT and J. L. MICHIELS, vol. **3** (London, 1960).

[4] G. BIORCI, A. FERRO and G. MONTALENTI: *Magnetic viscosity and annealing in magnetic field*. Technical Report 3a, A.R.D.C. Contract AF 61(514)-1331 (Nov. 1959).

[5] G. BIORCI, A. FERRO and G. MONTALENTI: to be published in *Phys. Rev.* (1960).

[6] H. PENDER and R. L. JONES: *Phys. Rev.*, **1**, 259 (1913).

[7] J. F. DILLINGER and R. M. BOZORTH: *Physics*, **6**, 279 (1935).

[8] J. F. DILLINGER and R. M. BOZORTH: *Physics*, **6**, 285 (1935).

[9] S. CHIKAZUMI: *Journ. Phys. Soc. Japan*, **5**, 327, 334 (1950).

[10] S. KAYA: *Rev. Mod. Phys.*, **25**, 49 (1953).

[11] L. NÉEL: *Journ. Phys. et Rad.*, **15**, 225 (1954).

[12] S. TANIGUCHI and T. OOMURA: *Sci. Rep. Ritu*, A **6**, 330 (1954).

[13] S. TANIGUCHI and M. YAMAMOTO: *Sci. Rep. Ritu*, A **7**, 269 (1955).

[14] S. TANIGUCHI: *Sci. Repts. Res. Inst. Tohoku Univ.*, Sez. AI, 269 (1955).

[15] R. M. BOZORTH: *A.I.E.E. Publication*, T-91, 69 (1957).

More recently other authors ([16-18]) have considered a new hypothesis to explain the annealing in a magnetic field of some alloys Fe–Co–Ni. They suggest that the effect is due to stacking faults associated with the presence of small percentage of oxigen in the alloy. These defects could be aligned by annealing in a magnetic field. The experiments in order to reveal these defects have been made with electron diffraction. In any case the possible presence of an effect of this type can be superposed to the other source of magnetic anisotropy *i.e.* to a directional short range ordering. On the other hand this type of effect can be considered of the same group as those due to precipitation of anisotropic particles and these effects will not be further considered in the present paper.

The aim of this work is to show that in any magnetic material it is sufficient, but not necessary, to have diffusion viscosity (due to interstitial atoms, to dislocations, to relaxation at the grain boundary, to rotation of couples) in order to have the possibility of annealing in a magnetic field. The energy of induced anisotropy K_u is nearly equal to the energy $H_t \cdot J_s$ associated to the viscosity field.

The anomaly of specific heat, associated to these processes of ordering, is so small as not to be observed experimentally. The experimental verifications, related in the following pages, are made for the case of iron containing interstitial carbon. Verifications on alloys Fe–Al, Fe–Si are reported in another work ([4]).

2. – Magnetic annealing in b.c.c. crystals with interstitial atoms.

The considerations of this section are qualitatively valid for b.c.c. structures with interstitial atoms. Qualitatively they are valid also for any structures and for any cause which produces magnetic viscosity. In another work ([4]) we have given the quantitative expressions for couples of solute atoms and for the different structures. Therefore it is necessary to prove that if a ferromagnetic material shows diffusion viscosity in a suitable temperature range it exhibits also magnetic annealing.

In the case of interstitial atoms the proof is straightforward: let us suppose that the interaction energy between an interstitial atom in a b.c.c. lattice and the magnetization vector be of the type $w \cos^2 \varphi$ (φ is defined according

([16]) E. A. NESBITT, H. J. WILLIAMS and R. M. BOZORTH: *Journ. Appl. Phys.*, **25**, 1014 (1954).

([17]) R. D. HEIDENREICH, E. A. NESBITT and R. D. BURBANK: *Journ. Appl. Phys.*, **30**, 995 (1959).

([18]) E. A. NESBITT and R. D. HEIDENREICH: *Journ. Appl. Phys.*, **30**, 1000 (1959).

to Fig. 1) and that the crystal is kept for long time at saturation in the direction α, β, γ. In this situation the theory of the diffusion viscosity given by NÉEL [1] shows that the distribution of the interstitial atoms in the lattice,
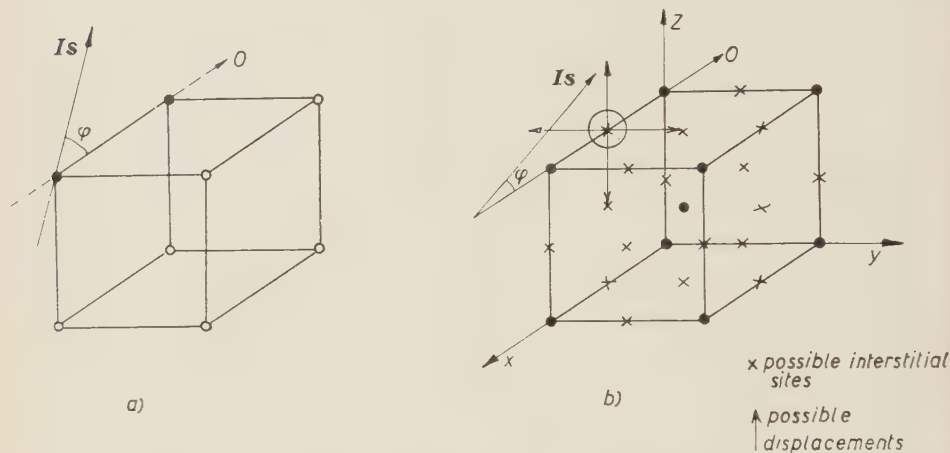

Fig. 1. – a) Interaction among couples of solute atoms and the magnetization vector. b) Positions of the interstitial atoms in a b.c.c. lattice.

when the diffusion has taken place, is such that to rotate suddenly the magnetization to a new direction α', β', γ' an energy E_d is required given by eq. (1)

$$(1) \qquad E_d = - W_0 \cdot (\alpha^2 \cdot \alpha'^2 + \beta^2 \cdot \beta'^2 + \gamma^2 \cdot \gamma'^2) \,.$$

From this property of the single domain NÉEL [4] deduces the existence of the diffusion viscosity both for the cases of 180° and 90° walls. The authors, in a previous note [19], have proved that also for a polycrystal the maximum value of the viscosity field is given by:

$$(2) \qquad H_t = \frac{W_0}{J_s} \,,$$

where W_0 is a quantity related [1] to the number of present interstitial atoms.

But we can note that, if we anneal the same specimen in a magnetic field, the whole specimen becomes a single domain and hence the energy E_d appears also macroscopically as an uniaxial anisotropy energy still given by eq. (1). Therefore a b.c.c. crystal with interstitial atoms should show both

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[19] G. BIORCI, A. FERRO and G. MONTALENTI: Atti Acc. Lincei, 5, 542 (1958).

magnetic viscosity and magnetic annealing. Let us consider (Fig. 1) an interstitial atom in a b.c.c. lattice and let q be the angle between the magnetization vector and the line joining the two atoms between which the interstitial atom is placed. As said above NÉEL [1] assumes that the interaction energy between the interstitial atom and J_s is $w \cdot \cos^2 q$. Because of this energy in each domain there is a small excess of interstitial atoms along the direction J_s. By demagnetizing the specimen, the Bloch walls will take a quite random arrangement, and domains will be formed in which the interstitial atoms will be equiparted along the three directions. Hence, immediately after demagnetization, the Bloch walls move as if there were no interstitial atoms. Because of the diffusion, as the time elapses, a small excess (or defect according to the sign of w) of atoms will reach the positions along J_s and therefore an additional work is required to rotate J_s by $90°$ when the walls move. In other words, long time after the demagnetization, in domains separated by $90°$ walls and within the wall thickness for the walls at $180°$ a uniaxial anisotropy energy per unit volume is produced locally in the crystallographic direction closest to J_s.

When the diffusion occurs during annealing in a magnetic field with the specimen at saturation, the direction of J_s is the same everywhere and hence the direction of the minimum of anisotropy, being in each grain along the (100) axis closest to J_s, is approximate the same all over the specimen.

Measuring the viscosity field means to measure the anisotropy energy induced by the diffusing interstitial atoms in each domain i.e. it is means to study the effect of the anisotropy on the wall movement. Taking a measure of induced anisotropy after annealing in a magnetic field means to examine the same effect when the direction of minimum induced anisotropy is approximately the same in the whole specimen.

The effects produced by the demagnetization in the case of diffusion viscosity and in the case of magnetic annealing are quite different. In the first case, assuming the anisotropy energy to be not too large with respect to the other restoring forces on the Bloch walls, immediately after demagnetization, the walls will fall in new random positions and they will be the boundaries of regions in which on the average the anisotropy energy difference is zero everywhere in the walls and in the domains. However, when the diffusion has taken place, the walls will separate regions not having statistically the same number of interstitial atoms along the three directions.

On the contrary when the induced anisotropy has grown up in the material at saturation and the demagnetization is made at lower temperature, whatever is the wall arrangement after demagnetization, the walls will separate regions having the same distribution of interstitial atoms. Shortly the demagnetization can cancel, by averaging, the anisotropy energy formed in each single domain and wall when the material is not annealed in magnetic

field under the only condition that the induced anisotropy energy is not too large with respect to the wall energy. The same operation, *i.e.* a demagnetization, does not cancel at all the induced anisotropy when this energy has grown up in the material at saturation.

Now we have to deduce the relation between eq. (2) and the energy of anisotropy K_u induced by annealing in a magnetic field. The K_u, for a polycrystal, is nothing but the value of E_d given by eq. (1) when the intensity of magnetization J_s has the same direction in the whole specimen. In order to calculate this average value it is sufficient to consider a system of reference axes fixed with the polycrystal, and another system of axes, variable from grain to grain, and coinciding with the crystallographic axes in each grain. (Eq. (1) is valid if it is referred to a system of axes coinciding with the crystallographic directions.) Then we have to write eq. (1) for one grain, but referred to the fixed axes, and compute the average value when the system of axes coinciding with the crystallographic directions takes all the possible orientations.

The problem is solved by the Euler formulas of the transformations of axes, or more rapidly by a method like the. one given by GALISSOT and VERGNE [20].

The obtained value is the following:

$$K_u = \tfrac{2}{5} W_0 .$$

Hence from eq. (2) we have:

(3) $H_t \cdot J_s = 2.5 \cdot K_u .$

In another work [1] we give the expression (3) for different structures when the energy of anisotropy is due to rotation of couples of solute atoms. The values of the numerical coefficients of eq. (3) remain all included between 1 and 2.5.

3. – Spicific heat and entropy anomalies due to interstitial atoms diffusing in a ferromagnetic lattice.

Let us consider a b.c.c. lattice and let us take a system of axes coinciding with the cube edges (Fig. 1). The interstitial atoms occupy octahedral sites at the middle of edges or at the center of faces, along x, or y, or z axes. We shall say that an interstitial atom is in an x (or y, or z) site when the line joining the two nearest neighbours is parallel to the x (or y, or z) axis.

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[20] F. GALISSOT and R. VERGNE: *Compt. Rend.*, **248**, 703 (1959).

Let the direction of J_s coincide with the x-axis. As well known from the diffusion viscosity theory ([1,21,22]) an interstitial atom in the x site has the energy w, while an interstitial atom in y or z sites has zero energy, the y and z directions being perpendicular to J_s: Therefore the energy of an interstitial atom can be either w or nought. Let N_T indicate the total number of interstitial atoms, N_1 the number of those in the upper level, and finally let N_0 indicate the number of interstitial atoms in the lower energy level. The ratio of the populations in the two levels is as well known given by

$$(4) \qquad \frac{N_1}{N_0} = \exp\left[-w/kT\right]$$

and as $N_T = N_0 + N_1$

$$(5) \qquad N_1 = \frac{N_T}{1 + \exp\left[w/kT\right]} .$$

The change of internal energy of the system with respect to an isotropic distribution is

$$(6) \qquad U = N_1 \cdot w,$$

where w is the maximum value of the interaction energy between the interstitial atom and J_s; k is the Boltzmann constant and T the absolute temperature.

Hence the contribution C_v to specific heat at constant volume (the diffusion of atoms from x to y or z sites occurs at constant volume with very good approximation) and referring to one gram mole of solute is given by:

$$(7) \qquad C_v = \frac{dU}{dT} c_a = kNc_a \frac{(w/kT)^2 \cdot \exp\left[w/kT\right]}{(1 + \exp\left[w/kT\right])^2},$$

where c_a is the atomic concentration of solute and N the Avogadro number.

The relationship (7) has a maximum for $w = kT$.

We may apply the previous relationship to the case of carbon atoms diffusing in α iron. The energy gap between the two levels, $i.e.$ the energy difference between a C atom in an x site and another one is the y site (Fig. 1),

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[21] G. BIORCI, A. FERRO and G. MONTALENTI: A.R.D.C. Contract no. AF 61(513)1331, Technical Note 1a (1958).

[22] G. BIORCI, A. FERRO and G. MONTALENTI: A.R.D.C. Contract no. AF 61(514)1331, Technical Note 2a (1958).

is given by $(^{1,21\text{-}23})$:

$$w = 6 \cdot 10^{-16} \text{ erg/atom} .$$

Furthermore the atomic concentration c_a is about $5 \cdot 10^{-4}$.

As stated, eq. (7) has a maximum for $w = kT$ *i.e.* in this case for $T = 4$ °K. By introducing these values in eq. (7) and recalling that $Nk = R = 1.98$ cal/g mole, we have:

(8) $C_v = 10^{-4}$ cal/g·mole .

Hence the anomaly of specific heat, also in the region of maximum effect, appears to be very low because the number of atoms which contribute to it is also small.

Considering the diffusion constants of carbon in α Fe it is obvious that the thermodynamical equilibrium at such a low temperature (4 °K) cannot be attained practically and therefore this specific heat anomaly cannot be observed. At those temperatures where the diffusion is rather rapid the value of C_v is even smaller than that given by eq. (8). For instance for $T = 300$ °K we have:

$$C_v = 5 \cdot 10^{-9} \text{ cal/mole} .$$

These values are extremely low with respect to the anomaly produced by a conventional order-disorder phenomenon. In this last case $C_v = 10$ cal/g·mole.

In a similar way it can be shown that in our case also the entropy anomaly is very small.

One can easily show that analogous conclusions are valid also when the magnetic viscosity is due to diffusion of couples of atoms, dislocations, or to grain boundary slip.

4. – Experimental results.

Since we had not a torsion balance the measurement of the induced anisotropy energy has been done with methods which may only give approximate results. In a first case we have measured the area included between the upper parts of the hysteresis loops drawn before and after the annealing in a magnetic field $(^{24,25})$. In another case we have measured the area included between the two curves of magnetization taken respectively: the first with the field of

$(^{23})$ G. DE VRIES, D. W. VAN GEEST, R. GERSDORF and G. W. RATHENAU: *Physica*, **25**, 1131 (1959).

$(^{24})$ R. M. BOZORTH: *Ferromagnetism* (New York, 1951), p. 625.

$(^{25})$ W. KÖSTER: *Theorie des Ferromagnetismus* (Berlin, 1956), p. 160.

measurement in the same direction of the annealing field, the second in the normal direction [26]. It is obvious that these methods are somewhat objectionable. It is also necessary to observe that the viscosity field is measured at temperature higher by about 50 °C than the energy of anisotropy. FERGUSSON [27] has verified that the induced anisotropy energy decreases rapidly with temperature. Moreover the viscosity field measured may be somewhat lower than the maximum attainable value because experimentally it is difficult to cover the whole spectrum of relaxation times.

Hence for these reasons the induced anisotropy energy deduced from the viscosity field will be generally lower than the one deduced from the magnetization curves.

The experimental results can be summarized as follows:

1) A strip of electrolytic iron about 0.25 mm thick, 40 mm wide 500 mm long was used. Carbon was introduced by keeping the specimen for 8 hour at 700 °C in H_2 with 10% Co and quenching in water. The amount of dissolved carbon measured by internal fraction methods was about 0.01% in
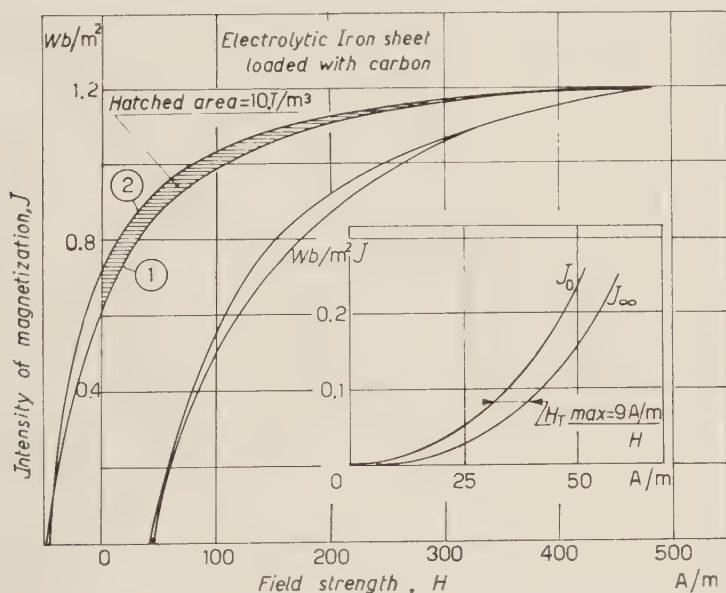

Fig. 2. – Electrolytic iron specimen loaded with interstitial carbon: 1) normal loop at – 70 °C: 2) loop obtained after cooling in magnetic field from – 16 C to – 70 °C in about two hours. The hatched area is equal to 100 erg/cm³. The maximum viscosity field is 9 A/m.

[26] S. CHIKAZUMI and T. OOMURA: Journ. Phys. Soc. Japan, **10**, 842 (1955).

[27] E. T. FERGUSSON: Congrès International sur la Physique de l'Etat Solide (Bruxelles, 2 June 1958), to be published.

weight. The measurements were taken with a permeameter and a ballistic galvanometer.

The maximum value of the viscosity field H_t, taken at $-16\ °C$, as shown in Fig. 2, is 9 A/m. Hence from (3), we have that the induced anisotropy constant K_u of the polycrystal is given by:

$$K_u = \frac{H_t J_s}{2.5}\ .$$

From the value of $H_t = 9$ A/m we obtain easily

$$K_u = 75\ \text{erg/cm}^3\ .$$

The hysteresis loops of the material were determined at a temperature of $-70\ °C$ with a maximum magnetization value of 1.2 Wb/m², respectively after cooling with no field applied and cooling in field. The cooling was performed in about two hours. The form of the loop obtained after cooling in magnetic field appears to be definitely more square.

The estimate of the induced anisotropy energy in the specimen was obtained by measuring the area between the upper parts of the two loops. From Fig. 2 we get:

$$K_u = 100\ \text{erg/cm}^3\ .$$

It shall be noted however that a higher value would have been obtained considering the difference between the curves relative to saturation loops of the specimen annealed in field in two orthogonal directions. However, for the reasons discussed previously this value results already noticeably higher than the value of the anisotropy energy deduced from the viscosity field give above.

2) A ring of Armco iron of the following dimensions was used: outside diameter 35 mm, inside diameter 28 mm, height 7 mm, in sheets of 1 mm. This specimen received the same treatment as the specimen in experiment 1), both as regards introduction of carbon and annealing in magnetic field. The carbon content in solid solution after the heat treatment was somewhat lower then in the preceding case and the viscosity field at $-16\ °C$ turned out to be 6.5 A/m (Fig. 3.).

From this value of the viscosity field we can again deduce from (3) the value of K_u

$$K_u = H_t \cdot J_s / 2.5 = 56\ \text{erg/cm}^3\ .$$

On this specimen two magnetization curves were determined at $-70\ °C$, after cooling in a magnetic field from $-16\ °C$, respectively one with a strong

field applied in the circumferential direction and one with field applied normal to it. The two curves are shown in Fig. 3: The effect of annealing in field appears to be rather strong. The area between the two curves (see Fig. 3),
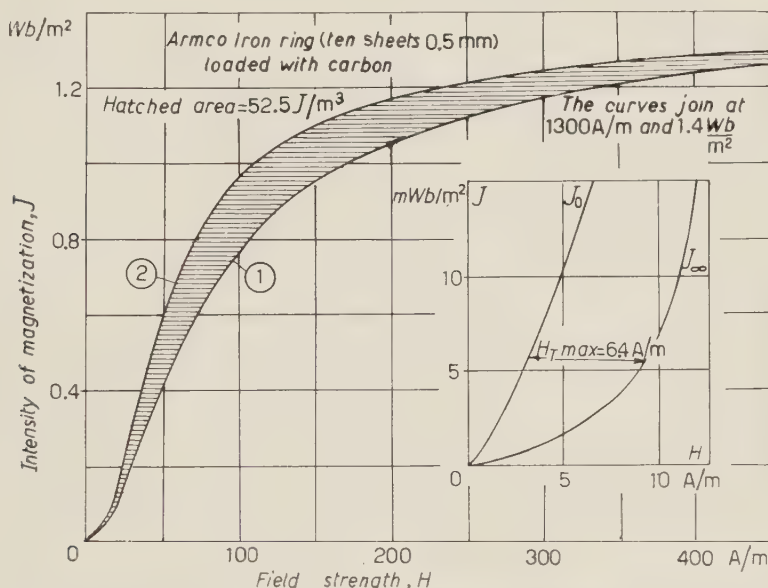

Fig. 3. – Annealed Armco iron ring loaded with interstitial carbon: ① magnetization curve after cooling to — 70 °C in a magnetic field in a direction perpendicular to the measuring field; ② magnetization curve after cooling to — 70 °C in a direction parallel to the measuring field. The hatched area is equal to 525 erg/cm³. The maximum viscosity field is 6.4 A/m.

as said above, corresponds to the anisotropy energy K_u induced by the annealing in magnetic field. This area turns out to be about 525 erg/cm³. This value is about one order of magnitude larger than the anisotropy energy deduced from the viscosity field given above. In this case however, apart from the other reasons already discussed, a contribution to K_u due to some change in the energy lost in the irreversible processes is to be expected.

Contemporaneously to our experiments other similar ones have been independently done by GRAHAM [28]. This author finds, in different experiments, induced anisotropy energy values between 60 erg/cm³ and 200 erg/cm³. In these experiments however the quantities of interstitial carbon in iron are not measured and therefore it is not possible to make a comparison with the energy associated to the viscosity field.

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[28] C. D. GRAHAM jr.: Magnetic properties of metals and alloys, in Amer. Soc. for Metal (1959), p. 288.

5. – Conclusion.

These results show beyond doubt that strong effects of annealing in a magnetic field are present at low temperatures corresponding to the diffusion viscosity due to interstitial atoms in solution (Fig. 2-3).

The value K_u of the anisotropy energy induced by annealing in a magnetic field measured by the changes in the magnetization curves appears to be larger than the corresponding value $H_t \cdot J_s/2.5$ deduced from the viscosity field. However the values deduced from H_t are always inferior to the expected values for the reasons already discussed and, on the other hand, the value of anisotropy energy estimated by the methods used is not very precise.

Finally, if we recall the experiments of BRISSONNEAU ([29]) on waps-waisted loops in iron with interstitial carbon atoms, we may conclude that when interstitial atoms are involved, viscosity field, squaring of the loops due to annealing in a magnetic field, and waps-waisted loops in the Rayleigh zone are all different aspects of the same phenomenon.

Besides it is experimentally verified that from a measurement of the viscosity field it is possible to calculate the energy of anisotropy which can be induced by annealing in a magnetic field. Moreover it is shown that these phenomena of ordering give variations of specific heat thousands of time less than the ones associated to the conventional order-disorder transformations present for instance in AB alloys.

([29]) P. BRISSONEAU: *Journ. Appl. Phys.*, **29**, 249 (1958).

RIASSUNTO

Nel presente lavoro si dimostra che condizione sufficiente ma non necessaria perchè si possa avere energia di anisotropia indotta da ricottura in campo magnetico è che il materiale presenti viscosità magnetica di diffusione. Se H_t è il valore massimo del campo magnetico di viscosità, il prodotto $H_t \cdot J_s$ è pressapoco eguale all'energia di anisotropia K_u indotta da ricottura in campo magnetico. Le anomalie di calore specifico associate al processo di ricottura in campo magnetico sono così modeste che non possono essere osservate sperimentalmente. Le verifiche sperimentali sono state eseguite su ferro contenente carbonio allo stato interstiziale. L'accordo si può considerare soddisfacente.

On the Relation between Zener Breakdown
and Residual Resistance in Crystals.

P. Gosar

Nuclear Institute « J. Stefan » - Ljubljana

Summary. — The theory of the Zener breakdown of crystals is extended
to the case of very strong electric fields. In the domain of breakdown
the resistivity of the crystal decreases very rapidly with the increase
of the electric field until it reaches a finite value, which depends no more
on the electric field. This resistivity is due to the scattering of electrons
by the periodic potential of the lattice. It is shown that the limit value of
resistivity can be calculated similarly as the residual resistance due to
impurities. The interpretation of residual resistance as the Zener break-
down of a crystal with many narrow energy gaps is given. The gaps in
the conduction band of the ideal crystal are produced by the presence
of impurity centers. A theory of the electron transport in the crystal
with many energy gaps is formulated. The comparison with the Boltzmann-
Bloch transport theory is given.

1. – Introduction.

In insulators and semiconductors we observe, for strong electric fields,
direct transitions of electrons from the valence band to the conduction band.
These transitions cause in some cases the electrical breakdown of insulators
and semiconductors. The theoretical explanation of the breakdown was first
given by Zener [1] who set up the theory for a one-dimensional crystal lattice.
The theory was improved and extended to three-dimensional lattices by

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[1] C. Zener: *Proc. Roy. Soc.*, A **145**, 523 (1934).

FRANZ (^{2,3}), HOUSTON (^{4}), McAFEE *et al.* (^{5}), HOMILIUS and FRANZ (^{6}), and recently by KELDYSH (^{7}) and KANE (^{8}).

In the domain of breakdown the resistivity of a crystal decreases very rapidly with the increase of the electric field. But also in the limit of extremely strong fields the crystal retains a finite resistance. This resistance does not depend on the electric field and is due, as will be shown, to the scattering of electrons by the periodic potential of the lattice. The limit value of resistivity can be calculated similarly as the residual resistance due to impurities. On the other hand we can interpret the residual resistance as a Zener breakdown of a crystal with many very narrow energy gaps. These are produced in the conduction band of the ideal crystal owing to the presence of impurity centers.

We shall see that the resistivity of a crystal with periodically distributed impurity centers is essentially the same as in the case of a random distribution. This statement is only correct if the electric field is strong enough for the appearance of a complete Zener breakdown. Such a breakdown is possible for small electric fields only if the energy gaps are very small. This implies a great distance between impurity centers or a small perturbation potential.

In the following we first extend the theory of the Zener breakdown to a case of very strong electric fields, where the existent theories are no more applicable. In the theories quoted above it is usually assumed that only a very small number of electrons pass from the valence to the conduction band. For a very strong electric field, however, this is no more the case. Secondly the transport theory for a crystal with many narrow energy gaps is formulated. A comparison with the Boltzmann-Bloch transport theory is given, as well.

2. – Zener formula for extremely strong electric field.

The following calculation is based essentially on Houston's method and Zener's calculations of a predissociation of gas molecules (^{9}).

The steady state wave functions for electrons in a crystal have the form

(^{2}) W. FRANZ: *Zeits. Phys.*, **113**, 607 (1939).

(^{3}) W. FRANZ: *International Conference on Semiconductors, Garmisch-Partenkirchen* 1956 (New York, 1958), p. 317.

(^{4}) W. V. HOUSTON: *Phys. Rev.*, **57**, 184 (1940).

(^{5}) K. B. McAFEE, E. J. RYDER, W. SHOCKLEY and M. SPARKS: *Phys. Rev.*, **83**, 650 (1951).

(^{6}) J. HOMILIUS and W. FRANZ: *Zeits. Naturfor.*, **9a**, 205 (1954).

(^{7}) L. V. KELDYSH: *Žurn. Èksp. Teor. Fiz.*, **33**, 994 (1957).

(^{8}) E. O. KANE: *Journ. Phys. Chem. Sol.*, **12**, 181 (1960).

(^{9}) C. ZENER: *Proc. Roy. Soc.*, A **137**, 696 (1932).

of Bloch functions

$$\psi_n(\boldsymbol{k}, \boldsymbol{r}) = u_n(\boldsymbol{k}, r) \exp [i\boldsymbol{k}\boldsymbol{r}] , \tag{1}$$

where $u_n(\boldsymbol{k}, \boldsymbol{r})$ is a periodic function of the unit cell of the crystal lattice and $\boldsymbol{k}$ the wave vector. For each value of $\boldsymbol{k}$ we find a series of eigenfunctions $\psi_n(\boldsymbol{k}, \boldsymbol{r})$ and eigenvalues $E_n(\boldsymbol{k}, \boldsymbol{r})$ which belong to different energy bands labelled by index n.

The application of an external electric field $\boldsymbol{F}$ to the crystal produces the perturbation $H' = + e\boldsymbol{F}\boldsymbol{r}$ of the crystal Hamiltonian H. Houston expands the solution $\Psi(t, \boldsymbol{r})$ of the perturbed time dependent Schrödinger equation in terms of $\psi_n[\boldsymbol{k}_1 - (e\boldsymbol{F}/\hbar)t, \boldsymbol{r}]$, where $\boldsymbol{k}_1$ is the wave vector of the initial state of a particular electron at the moment $(t = 0)$ of the application of the electric field to the crystal. Thus he obtains a system of differential equations for the expansion coefficients $c_n(t)$. Only the transitions between two adjacent bands are of interest. We label these bands by $n = 1$ and $n = 2$. The system of equations is usually solved with the assumption that only few electrons from the band of lower energy (the valence band, $n = 1$) can pass over to the other band (the conduction band, $n = 2$), when the electric field is applied to the crystal. This is true for relatively small fields only. In such cases we take $c_1(t) = 1$.

The transitions between two bands are only possible if the matrix element $X_{12} = i \int u_1^*(\boldsymbol{r}) \operatorname{grad}_k u_2(\boldsymbol{r}) \, d\tau$ is different from zero. Here the integration is performed over the volume of the unit cell. This means that the character of functions u_1 and u_2 changes with vector $\boldsymbol{k}$. In the following we shall assume that u_1 and u_2 are linear combinations of two functions, which vary only slowly with $\boldsymbol{k}$. The principal variation of u_1 and u_2 is produced by the variation of the coefficients of linear combinations.

By analogy with Zener's calculations of a predissociation of gas molecules we introduce a new pair of base functions $\Phi_1(\boldsymbol{k}, \boldsymbol{r})$ and $\Phi_2(\boldsymbol{k}, \boldsymbol{r})$ defined as

$$\begin{cases} \Phi_1(\boldsymbol{k}, \boldsymbol{r}) = \alpha_{11}(\boldsymbol{k}) \, \psi_1(\boldsymbol{k}, \boldsymbol{r}) + \alpha_{12}(\boldsymbol{k}) \, \psi_2(\boldsymbol{k}, \boldsymbol{r}) , \\ \Phi_2(\boldsymbol{k}, \boldsymbol{r}) = \alpha_{21}(\boldsymbol{k}) \, \psi_1(\boldsymbol{k}, \boldsymbol{r}) + \alpha_{22}(\boldsymbol{k}) \, \psi_2(\boldsymbol{k}, \boldsymbol{r}) . \end{cases} \tag{2}$$

The coefficients $\alpha_{ik}(\boldsymbol{k})$ are still undetermined functions of $\boldsymbol{k}$, but restricted by the requirement that Φ_1 and Φ_2 are mutually orthogonal and normalized in the unit cell. The expansion of $\Psi(t, \boldsymbol{r})$ by new base functions reads

$$\Psi(t, \boldsymbol{r}) = a_1(t) \, \Phi_1 \left(\boldsymbol{k}_1 - \frac{e\boldsymbol{F}}{\hbar} t, \boldsymbol{r} \right) + a_2(t) \, \Phi_2 \left(\boldsymbol{k}_1 - \frac{e\boldsymbol{F}}{\hbar} t, \boldsymbol{r} \right), \tag{3}$$

where $a_1(t)$ and $a_2(t)$ are coefficients of expansion. The new basic functions are

no more solutions of the unperturbed Schrödinger equation. Instead of this we have

(4) $$H\Phi_i = f_{i1}\Phi_1 + f_{i2}\Phi_2 ; \qquad\qquad i = 1, 2$$

with

(5) $$f_{ik}(\boldsymbol{k}) = \alpha_{i1}(\boldsymbol{k})\,\alpha_{k1}^*(\boldsymbol{k})\,E_1(\boldsymbol{k}) + \alpha_{i2}(\boldsymbol{k})\,\alpha_{k2}^*(\boldsymbol{k})\,E_2(\boldsymbol{k}) .$$

From (5) we derive the coefficients α_{ik}. All α_{ik} are to be chosen real in order to assure real values of f_{ik}. Thus

(6) $$\begin{cases} \alpha_{11} = +\sqrt{x}\,, & \alpha_{12} = +\sqrt{1-x}\,, \\ \alpha_{21} = -\sqrt{1-x}\,, & \alpha_{22} = +\sqrt{x}\,, \end{cases}$$

with

(7) $$x = \frac{1}{2}\left[1 + \frac{f_{11}(\boldsymbol{k}) - f_{22}(\boldsymbol{k})}{E_1(\boldsymbol{k}) - E_2(\boldsymbol{k})}\right].$$

Further it follows from the definition of f_{ik} that

(8) $$\begin{cases} E_1(\boldsymbol{k}) = \dfrac{f_{11}(\boldsymbol{k}) + f_{22}(\boldsymbol{k})}{2} - \sqrt{f_{12}^2(\boldsymbol{k}) + \left[\dfrac{f_{11}(\boldsymbol{k}) - f_{22}(\boldsymbol{k})}{2}\right]^2}\,, \\[2em] E_2(\boldsymbol{k}) = \dfrac{f_{11}(\boldsymbol{k}) + f_{22}(\boldsymbol{k})}{2} + \sqrt{f_{12}^2(\boldsymbol{k}) + \left[\dfrac{f_{11}(\boldsymbol{k}) - f_{22}(\boldsymbol{k})}{2}\right]^2}\,. \end{cases}$$

In the neighbourhood of a point $\boldsymbol{k}_0$, where $E_2 - E_1$ has its minimum value on a chosen line $\boldsymbol{k}_1 - (e\boldsymbol{F}/h)t$, the energy surfaces can be in many cases approximated by the assumption that f_{12} is a constant and f_{11} and f_{22} are linear functions of $(\boldsymbol{k} - \boldsymbol{k}_0)$. Therefore we put

(9) $$\begin{cases} f_{11}(\boldsymbol{k}) = c + \boldsymbol{a}(\boldsymbol{k} - \boldsymbol{k}_0)\,, \\ f_{22}(\boldsymbol{k}) = c - \boldsymbol{b}(\boldsymbol{k} - \boldsymbol{k}_0)\,, \end{cases}$$

where c is a constant and $\boldsymbol{a}$, $\boldsymbol{b}$ are vectors. $2f_{12}$ is equal to the energy gap $\Delta E(\boldsymbol{k}_0) = E_2(\boldsymbol{k}_0) - E_1(\boldsymbol{k}_0)$.

The calculation of probability for the interband transitions now reduces to the essentially equivalent calculation of probability for transitions from the state described by Φ_1 to the state described by Φ_2. For great values of $(\boldsymbol{k} - \boldsymbol{k}_0)$ the functions Φ_1 and Φ_2 are essentially equal to $\psi_1(\boldsymbol{k}, \boldsymbol{r})$ and $\psi_2(\boldsymbol{k}, \boldsymbol{r})$ respectively or viceversa.

Using the expansion (3) we obtain from the time dependent Schrödinger

equation the following differential equations for $a_1(t)$ and $a_2(t)$.

(10)
$$\begin{cases} i\hbar \dfrac{\mathrm{d}a_1}{\mathrm{d}t} = [c + \boldsymbol{a}(\boldsymbol{k} - \boldsymbol{k}_0)] \, a_1 + f_{12}\, a_2 \,, \\[2ex] i\hbar \dfrac{\mathrm{d}a_2}{\mathrm{d}t} = f_{12}\, a_1 + [c - \boldsymbol{b}(\boldsymbol{k} - \boldsymbol{k}_0)] \, a_2 \,, \end{cases}$$

where $\boldsymbol{k}$ is a function of time; $\boldsymbol{k} - \boldsymbol{k}_1 - (eF/h)t$. Here we neglected the time variation of $\varPhi_1(\boldsymbol{k}, \boldsymbol{r}) \exp[-i\boldsymbol{k}\boldsymbol{r}]$ and $\varPhi_2(\boldsymbol{k}, \boldsymbol{r}) \exp[-i\boldsymbol{k}\boldsymbol{r}]$.

Now we introduce new unknowns A_1, A_2 in the place of a_1, a_2 by means of

(11)
$$\begin{cases} A_1(t) = a_1(t) \exp\left[+\dfrac{i}{\hbar} \int_0^t [c + \boldsymbol{a}(\boldsymbol{k} - \boldsymbol{k}_0)] \mathrm{d}t \right], \\[3ex] A_2(t) = a_2(t) \exp\left[+\dfrac{i}{\hbar} \int_0^t [c - \boldsymbol{b}(\boldsymbol{k} - \boldsymbol{k}_0)] \, \mathrm{d}t \right]. \end{cases}$$

Hence

(12)
$$\begin{cases} i\hbar \dfrac{\mathrm{d}A_1}{\mathrm{d}t} = f_{12} A_2 \exp\left[+\dfrac{i}{\hbar} \int_0^t (\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{k} - \boldsymbol{k}_0) \, \mathrm{d}t \right], \\[3ex] i\hbar \dfrac{\mathrm{d}A_2}{\mathrm{d}t} = f_{12} A_1 \exp\left[-\dfrac{i}{\hbar} \int_0^t (\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{k} - \boldsymbol{k}_0) \, \mathrm{d}t \right]. \end{cases}$$

By the elimination of $A_2(t)$ we obtain the following differential equation for $A_1(t)$

(13)
$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} - \frac{i}{\hbar}(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{k} - \boldsymbol{k}_0) \frac{\mathrm{d}A_1}{\mathrm{d}t} + \frac{f_{12}^2}{\hbar^2} A_1 = 0 \,.$$

The solutions of (13) can be expressed by Weber functions of complex argument.

The probability w of the transition from the valence to the conduction band for a single electron is obtained from the asymptotic expression of A_1 for $t - t_0 = +\infty$, where t_0 is the time of the passage of the electron by the point $\boldsymbol{k}_0$. Thus

(14)
$$w = \left| \frac{A_1(t_0 + \infty)}{A_1(t_0 - \infty)} \right|^2 = \exp\left[-\frac{2\pi f_{12}^2}{e(\boldsymbol{a} + \boldsymbol{b})E} \right]; \qquad |A_1(t_0 - \infty)| = 1 \,.$$

Instead of $|\boldsymbol{a} + \boldsymbol{b}|$ we introduce in (14) the effective mass m^* for the direction

of vector $\boldsymbol{a}+\boldsymbol{b}$. We obtain

(15)
$$w = \exp\left[-\frac{\pi\sqrt{2m^*}\,(\Delta E)^{\frac{3}{2}}}{4\hbar e F \cos\alpha}\right],$$

where α is the angle between the directions of vectors $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{F}$.

For the one dimensional case the formula (15) is identical with the expression derived by McAfee *et al*. The formula (15) claims its validity also for very strong electric fields. The upper limit for fields is given by the condition that the asymptotic expansion should be admissible. Thus we obtain the following restriction

(16)
$$\frac{|(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{F}|\,|\Delta\boldsymbol{k}|^2}{eF^2} \gg 1 ,$$

where $|\Delta\boldsymbol{k}|$ is the distance in the space $\boldsymbol{k}$ between two successive points of minimum interband energy difference.

3. – Strong electric field conductivity of crystals with small energy gaps.

In this Section we shall consider the influence of interband transitions on electric conductivity. We shall discuss the case of a crystal with many not overlapping energy bands. We assume that many of them are partially occupied by electrons at a given temperature.

We begin with the evaluation of the current density $\boldsymbol{i}(\boldsymbol{k}_0)$ of electrons across the surface defined by vectors $\boldsymbol{k}_0$. This surface usually represents the Brillouin zone boundary. First we assume that the lower band is only partially occupied and the upper one completely empty. The concentration of electrons, which have the wave vector $\boldsymbol{k}$ in the interval $\mathrm{d}\boldsymbol{k}$ about the value $\boldsymbol{k}_0$ is $(1/4\pi^3)f_1(\boldsymbol{k}_0)\,\mathrm{d}\boldsymbol{k}$, where $f_1(\boldsymbol{k})$ is the distribution function for electrons in the lower band. The current density $\boldsymbol{i}(\boldsymbol{k}_0)$ for unit volume of crystal is now given by

(17)
$$\boldsymbol{i}(\boldsymbol{k}_0) = -f_1(\boldsymbol{k}_0)\,\frac{e\boldsymbol{F}}{4\pi^3\hbar}\,\exp\left[-\frac{\pi\sqrt{2m^*}\,(\Delta E)^{\frac{3}{2}}}{4\hbar e F \cos\alpha}\right].$$

For electric fields which are strong enough we can expand the exponential function into a series and retain only the first two terms. In such a case we obtain

(18)
$$\boldsymbol{i}(\boldsymbol{k}_0) = -f_1(\boldsymbol{k}_0)\,\frac{e\boldsymbol{F}}{4\pi^3\hbar} + f_1(\boldsymbol{k}_0)\,\frac{\sqrt{2m^*}\,(\Delta E)^{\frac{3}{2}}}{16\pi^2\hbar^2 \cos\alpha}\,\frac{\boldsymbol{F}}{|\boldsymbol{F}|} .$$

The second term in (18) gives the current density of reflected electrons at the

boundary surface. Our attention is drawn to the important fact that the current of reflected electrons does not depend on the electric field. We see that the resistivity due to the interband transitions must remain finite also for extremely strong electric fields.

There arise some complications in the deduction of expressions (17) or (18) if the upper band is not empty and the current density is great. In such a case we must also consider the transitions of electrons from the upper band to the lower one. Because of the exclusion principle the transition current increases as some states in the lower band become occupied by the electrons arriving from the upper band. The reflection current density is obtained by multiplying the last term of (18) by $[1 - f_2(\mathbf{k}_0)]$, where $f_2(\mathbf{k})$ is the distribution function for electrons in the upper band.

Now we shall deduce the transport equation for electrons in the crystal submitted to the electric field. Beside the interband transitions we also assume the isotropic scattering of electrons by phonons. Let τ be the relaxation time for the electron-phonon scattering. We denote the perturbed distribution function for the electrons in the band i, as above, by the symbol $f_i(\mathbf{k})$. Now the system of transport equations reads.

$$(19) \qquad -\frac{e\mathbf{F}}{4\pi^3\hbar}\nabla f_i(\mathbf{k}) + \frac{f_i(\mathbf{k}) - f_{0i}(\mathbf{k})}{4\pi^3\tau} + I_{i,i+1}(\mathbf{k}) - I_{i-1,i}(\mathbf{k}) = 0 \, ,$$

where $f_{0i}(\mathbf{k})$ is the unperturbed distribution function for electrons, *i.e.* the Fermi-Dirac function. $I_{i,i+1}(\mathbf{k})$ is the density of the electron current from the band i to the band $i+1$. We consider the function $I_{i,i+1}(\mathbf{k})$ as different from zero only in the neighbourhood of zone boundaries or surfaces defined by vectors $\mathbf{k}_0$.

Instead of performing all the calculations in the domain of the reduced zone we choose the general Brillouin pattern of zones. The bands i and $i+1$ lie now in the adjacent zones of the Brillouin pattern. Consequently $I_{i,i+1}(\mathbf{k})$ represents the current density across the boundary between the i-th and $(i+1)$-th zone from a given $\mathbf{k}$ in the zone i to the equivalent $\mathbf{k}'$ in the zone $i+1$.

In the case of a strong electric field the reflection current density is small and does not depend on the field. If also the distribution function $f(\mathbf{k})$ (the index i is here omitted) changes only little from one zone to another, we can replace the system (19) of equations by one integro-differential equation only

$$(20) \qquad -\frac{e\mathbf{F}}{\hbar}\nabla f(\mathbf{k}) + \frac{f(\mathbf{k}) - f_0(\mathbf{k})}{\tau} + \int \{ W(\mathbf{k}, \mathbf{k}') f(\mathbf{k})[1 - f(\mathbf{k}')] -$$
$$- W(\mathbf{k}', \mathbf{k}) f(\mathbf{k}')[1 - f(\mathbf{k})] \} \, d\sigma = 0 \, .$$

In this equation $W(\mathbf{k}, \mathbf{k}')$ is the probability function for the reflection of electrons on the zone boundaries from the direction $\mathbf{k}$ to the direction $\mathbf{k}'$.

The integration in the last term of (20) extends over the surface of the unit sphere. $d\sigma$ is the differential solid angle in the direction $\mathbf{k}'$. In all reflections the absolute value of $\mathbf{k}$ is preserved, *i.e.* $|\mathbf{k}'| = |\mathbf{k}|$.

We determine $W(\mathbf{k}, \mathbf{k}')$ from the current density of reflected electrons on the zone boundaries. Let us evaluate the number of electrons reflected in the volume element dV of space $\mathbf{k}$ on Fig. 1 to the directions $\mathbf{k}'$ in the solid angle $d\sigma$. We choose for dV a cylinder with a base S which is approximately parallel to zone boundaries producing reflections of electrons. The height of the cylinder is v. The volume dV extends over many zones. The zone boundaries are determined by the condition that $\mathbf{K} = \mathbf{k} - \mathbf{k}'$ is a lattice vector of the reciprocal lattice and by the postulation $|\mathbf{k}'| = |\mathbf{k}|$. The number of reflected electrons per unit time by the particular zone boundary j in the volume element dV is

Fig. 1.

$$I_j = f(\mathbf{k})[1 - f(\mathbf{k}')] \frac{S\sqrt{2m^*}(\Delta E)^{\frac{3}{2}}}{16\pi^2\hbar^2}. \tag{21}$$

The zone boundary j is completely determined by the reciprocal lattice vector $\mathbf{K}_j$. If we choose now other lattice vectors close to the vector $\mathbf{K}_j$ we obtain further boundary planes in the volume element dV. These planes also produce reflections of electrons in directions close to $\mathbf{k}'$. The number of reflected electrons per unit time is for all these planes approximately equal to (21). The number of possible planes depends on r and the solid angle $d\sigma$, which restricts the admissible directions of reflection. The volume dV_1 of space $\mathbf{k}$ in which the vectors $\mathbf{K}$ must lie is evidently

$$dV_1 = \frac{v}{\sin(\vartheta/2)}|\mathbf{K}_j|^2 d\sigma, \tag{22}$$

where ϑ is the angle between the directions of vectors $\mathbf{k}$ and $\mathbf{k}'$. Dividing dV_1 by the volume r_0 of the unit cell of the reciprocal lattice one obtains the number
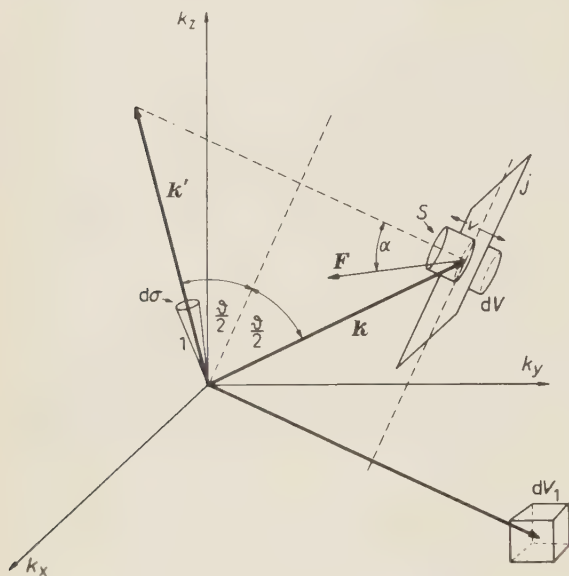
of zone boundaries which produce the reflections of electrons in dV to the directions in the cone $d\sigma$. Hence the total number I of reflected electrons per unit time in dV to the direction $\boldsymbol{k}'$ is

$$(23) \qquad I = f(\boldsymbol{k})\,[1 - f(\boldsymbol{k}')]\,\frac{\sqrt{2m^*}\,(\Delta E)^{\frac{3}{2}}}{16\pi^2\hbar^2 v_0 \sin(\vartheta/2)}\,|\boldsymbol{K}_i|^2\,dV\,d\sigma\,.$$

Comparing I with the last term in (20) we obtain, in using the relation $|\boldsymbol{K}_i| = 2\,|\boldsymbol{k}|\sin(\vartheta/2)$, that

$$(24) \qquad W(\boldsymbol{k},\,\boldsymbol{k}') = \frac{\pi\sqrt{2m^*}\,(\Delta E)^{\frac{3}{2}}}{v_c\hbar^2}\,|\boldsymbol{k}|^2\sin(\vartheta/2)\,.$$

Now, expressing in (24) m^* by ΔE and $|\boldsymbol{a}+\boldsymbol{b}|$, we find

$$(25) \qquad W(\boldsymbol{k},\,\boldsymbol{k}') = \frac{\pi(\Delta E)^2}{v_c\hbar\,|\boldsymbol{a}+\boldsymbol{b}|}\,|\boldsymbol{k}|^2\sin(\vartheta/2)\,.$$

Let, for instance, the energy of the electron in the unperturbed crystal be a quadratic function of the square of the unreduced wave vector $\boldsymbol{k}$, i.e.

$$(26) \qquad E(\boldsymbol{k}) = \frac{\hbar^2}{2m'}\,|\boldsymbol{k}|^2\,,$$

where m' is the effective mass of the electron. For such a case one obtains from (25) that

$$(27) \qquad W(\boldsymbol{k},\,\boldsymbol{k}') = \frac{\pi m'(\Delta E)^2}{2v_c\hbar^3}\,|\boldsymbol{k}|\,.$$

This formula is essentially equivalent to the probability function in the theory of electrical conduction for the elastic scattering of electrons by some perturbing potential $U(\boldsymbol{r})$, whose matrix element $U_{kk'}$ is equal to $\Delta E/2$. This is the case in Brillouin's approximation.

The time of relaxation due to interband transitions is now

$$(28) \qquad \tau = \int W(\boldsymbol{k},\,\boldsymbol{k}')(1 - \cos\vartheta)\,d\sigma\,.$$

Thus we have obtained an interesting relation between the Zener breakdown and the residual resistance. One can consider the residual resistance as due to a complete Zener breakdown.

4. – Conclusions.

It is known that the Boltzmann-Bloch equation is only applicable if the effective range of scattering potential is small compared with the distance

between neighbouring scattering centers and if $kT \gg h/\tau$. The theory of impurity conduction, which is based on Zener breakdown (as presented in this paper), imposes less rigorous conditions for its validity. The first restriction concerning the range of the scattering potential is no more necessary. The second condition $kT \gg h/\tau$ is replaced by the assumption that kT extends over many energy bands of the perturbed crystal. However, we have here an additional restriction referring to the value of the applied electric filed. If we have to deal with spherical energy surfaces of an unperturbed crystal, the value of the field must be such that

$$(29) \qquad \frac{\pi m'(\Delta E)^2}{4\hbar^2 |\boldsymbol{k}| |\Delta \boldsymbol{k}|} \ll \frac{\hbar}{\tau'} \ll \frac{2\hbar^2 |\boldsymbol{k}| |\Delta \boldsymbol{k}|}{m'} ,$$

where $\tau' - |\Delta \boldsymbol{k}| / |(\mathrm{d}\boldsymbol{k}/\mathrm{d}t)|$ is the time between two successive passages of a particular electron through the points of minimum interband energy difference. In (29) we assumed $\vartheta = 180°$ and $\boldsymbol{F}$ parallel to $\boldsymbol{a}+\boldsymbol{b}$.

The lower value of the field depends on energy gaps between successive bands. In impurity conductors the energy gaps in the conduction band depend essentially on the order in the distribution of impurity centers. For a random distribution they are smallest and increase with the order in the distribution. The gaps also increase with the concentration of impurities.

In the theory we completely neglected the collisions of electrons and phonons. We can expect that the lower value of the allowed fields will diminish owing to electron-phonon interactions or, in cases of very small gaps, disappear completely. We also neglected polarization effects, which are usually small for the states defined by the new base functions Φ_1 and Φ_2.

RIASSUNTO (*)

La teoria del collasso resistivo dei cristalli dovuta a Zener si estende al caso di campi elettrici molto forti. Nella zona di collasso la resistività del cristallo decresce molto rapidamente al crescere del campo elettrico sinché raggiunge un valore definito, che non dipende più dal campo. Questa resistività è dovuta allo scattering di elettroni da parte del potenziale periodico del reticolo. Si mostra come il valore limite della resistività possa essere calcolato in modo simile alla resistività dei conduttori con impurezze. Si dà l'interpretazione della conduttività per impurezze come collasso di Zener di un cristallo con molte strette discontinuità energetiche. Le discontinuità nella zona di conduzione di un cristallo ideale sono prodotte dalla presenza di centri di impurità. Si formula una teoria del trasporto di elettroni in un cristallo con molte discontinuità energetiche. Si fa il confronto con la teoria del trasporto di Boltzmann-Bloch.

Extensive Air Showers and the Upper Atmosphere.

C. A. PEARSON and S. T. BUTLER

The Daily Telegraph - Theoretical Department (*)
School of Physics, The University of Sydney - Sydney

Summary. — An attempt has been made to explain the solar variations in counting rate of extensive air showers, reported by McCusker, in terms of a tidal change in the height of the first interaction of the primary. It is possible to obtain the very large amplitudes McCusker has found with an amplitude of tidal motion in the ozone layer of between 3 and 10 km, and provided nearly all of the high energy primaries involved are heavy.

1. – Introduction.

During the past few years a number of groups [1] using various types of detectors have looked for changes in the rates of counting large air showers with solar time. Some of them [2,3] have found that their counting rate varies periodically with marked twelve hour and twenty-four hour components.

It is remarkable that where the showers are detected with M-units, these periodic variations are most firmly established and have the greatest amplitude.

The M-units consists of three small geiger counters of area 14 cm^2 at the corners of an equilateral triangle of side 20 cm. They are used in coincidence with an extensive geiger tray several metres away to screen out any local showers.

Because the counters are so small the units trigger on high particle densities which are to be found only near the core of large showers.

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[1] W. GALBRAITH: *Extensive Air Showers* (Buttherworth, 1958), Table (7.1).
[2] C. B. A. McCUSKER and B. G. WILSON: *Nuovo Cimento*, **10**, 188 (1956).
[3] C. B. A. McCUSKER, D. E. PAGE and R. A. REID: *Phys. Rev.*, **113**, 712 (1959).

Several groups working with McCUSKER ([2,3]) have averaged the bi-hourly counting rate over periods of about a year in Dublin, Jamaica and Sydney. In each case the counting rate shows marked diurnal and semi-diurnal variations which are strongly correlated with the pressure wave at ground. The amplitude of these fluctuations has varied from six to sixteen per cent of the total counting rate.

The primaries beginning the showers have energies greater than 10^{15} eV: and it seems extremely unlikely that their flux should show diurnal and semi-diurnal fluctuations.

For this reason McCUSKER ([3]) *et al.*, in discussing the results from Jamaica and Dublin for 1956, 1957, proposed that the variation in counting rate is caused by a change in the structure of the showers. The large amplitude suggested to them that this change in shower structure was associated with a tidal motion of the height of the first interaction, and that this interaction occurred in a region where tidal changes in its height could be very large.

Below 30 km it is possible to make balloon measurements of pressure, temperature and winds in the atmosphere, and it is known that diurnal and semi-diurnal pressure changes are small. However, it is possible that above this region large tidal motions occur.
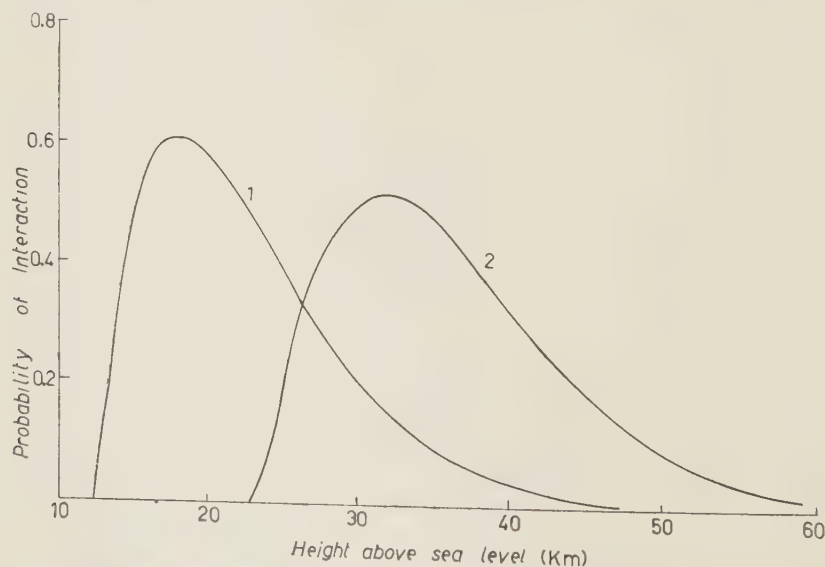

Fig. 1. – The probability that protons (curve 1) and iron nuclei (curve 2) will interact at various heights in the earth's atmosphere. The cross-sections for interaction were taken to be geometric and the air densities those of the 1952 Rocket Panel ([4]).

In Fig. 1 the probability of interaction is plotted as a function of height for protons and iron primaries assuming that the cross-section is geometric

and that nuclear radii are given by $r = 1.2 A^{\frac{1}{3}}$ in fermis. The densities are those adopted by the Rocket Panel [1].

It is most unlikely that the first interaction of proton primaries is in the region where large atmospheric tidal motions can occur. Evidence from air showers indicates that the cross-section for interaction of the high energy secondaries is substantially geometric.

Measurements of proton cross-sections at 860 MeV by CHEN [5] et al., indicate that the inelastic cross-section for protons in air is less than .25 barns. This is reported to be in close agreement with the value found by RADJOPADHYE [6] for protons interacting in emulsions. The cross-section needs to increase with energy by about ten times in order that nearly all the protons interact above 30 km.

Moreover, if the primary particles are protons it is difficult to see how a change in the height of the first interaction alters the density of electrons near to the core at ground level. It is known from shower theory that most of the electrons near to the core result from nuclear collisions of the high energy secondaries low down in the atmosphere. From the measurement of jets in emulsions it has been found that collisions between protons and nuclei become much more elastic as the energy increases and that at high energies most of the collision energy is concentrated in one nucleon secondary. It would seem that this secondary in its collisions lower down produces the nuclear active particles controlling the electron density near to the shower axis.

There is other evidence that many of these high energy air showers are not begun by protons. A number of authors [7,8] looking close to the shower axis have seen high energy penetrating particles, all with similar energies which appear to be fragments of the primary. Others [9,10] have observed sharp peaks in the electron density near to the shower core.

In nine showers from primaries between $3 \cdot 10^{15}$ and $2 \cdot 10^{16}$ eV energy, DMITRIEV et al. [7] have reported an average of more than ten nuclear active particles with energies greater than 10^{11} eV in each shower core, spread over a region extending several metres from the shower axis.

Recently PETERS has suggested that less than half the primaries with

[1] Rocket Panel: Phys. Rev., 88, 1027 (1952).
[5] F. F. CHEN, C. P. LEAVITT and A. M. SHAPIRO: Phys. Rev., 99, 857 (1955).
[6] V. Y. RAJOPADHYE and C. T. WADDINGTON: Phil. Mag., 3, 19 (1958).
[7] V. A. DMITRIEV, G. B. HRISTIANSEN and G. V. KULIKOV: Suppl. Nuovo Cimento, 8, 587 (1958).
[8] S. I. NIKOLSKIJ, U. N. VAVILOV and V. V. BATOV: Dokl. Ak. Nauk USSR., 111, 71 (1956).
[9] R. E. HERRIMAN and W. E. HAZEN: Phys. Rev., 90, 496 (1953).
[10] W. P. DAVIS, W. E. HAZEN and R. E. HERRIMAN: Nuovo Cimento, 12, 233 (1954).

a given total energy below 10^{15} eV are protons. He has indicated that the pro-
portion of heavy primaries should increase rapidly with energy above this.

In the following we will discuss the triggering of M-units by showers begun
by primaries containing at least several nucleons, at a level in the atmosphere
which suffers large tidal fluctuations.

The way in which the primary disintegrates on interaction (*) in the upper
atmosphere controls the shower structure close to the shower axis. As a rough
model for this disintegration we have assumed that a « fireball » is formed
about the primary which then decays isotropically in its own reference frame.
The nucleons emitted from the fireball with mean energies of 1 or 2 GeV, will
form separate shower cores. The spread of these cores at the ground will
determine the electron density near to the axis. It is to be expected that as
the height of the first collision increases, this spread will increase, lowering the
mean electron density and the counting rate.

The spread of the core is controlled by the energy with which the nucleons
are emitted in the F frame, and the Lorentz factor of this frame with respect
to the ground.

For a given energy the speed of the fireball in the laboratory system de-
pends on the size of the primary and on the nature of the collision. If the
primary is as big as iron it is possible that several hundred mesons are
formed in the F frame. Even if no momentum is transferred to the atmos-
pheric nucleus, that given to the mesons in the « fireball » is sufficient to re-
duce the Lorentz factor of the primary by several times. For if $\gamma_1 m_1$ be the
Lorentz factor and the mass of the primary before collision, $\beta_1^2 = 1 - \gamma_1^{-2}$, and
$\gamma_2 \beta_2$ be the corresponding quantities for the fireball, then

$$(1) \qquad \gamma_1 m_1 \beta_1 = m^* \beta_2 \gamma_2 ,$$

where $m^* c^2$ is the energy of the fireball.

It is probable than m^* increase with the primary energy. PETERS has
suggested that m_1 does also. If this is the case it is possible that the size of
the core stays relatively constant for all the shower sizes to which McCusker's
apparatus is sensitive. The importance of this will be discussed with the
results in Section **4**.

In Section **2** we have determined the distribution of the nucleon cores.

Since the steepness of the electron distribution for showers initiated by
nucleons fluctuates from shower to shower near to the core and is not well
known, we have chosen for it a distribution whose steepness can be adjusted.

(*) It can be shown that even for the high energies involved the initial disin-
tegration of a heavy primary must be produced by a direct nuclear collision, with little
contribution from « Coulomb dissociation » collisions.

In Section **3** the electron distribution from a heavy primary is determined in terms of it.

In Section **4** the counting rate of an M-unit is found from the electron distribution and the primary energy spectrum.

A number of calculations of the counting rate for showers starting at different heights have been made using « Silliac », in order to compare with the amplitudes found by McCUSKER. The results of these calculations, and the effects of various assumptions about the first interaction, the primary spectrum and the distribution about single cores are discussed in Section **5**.

2. – The distribution of cores.

In determining the distribution of nucleon cores we have assumed that the « fireball » engulfs the whole primary and that from it the nucleons are emitted independently. That is, we have taken an extreme case where the collision produces no heavy fragments.

For a nucleon emitted at an angle θ to the direction of motion of the primary, with momentum p and energy E in the F system, and with corresponding quantities θ', p', E' in the laboratory system it is easy to obtain the relation

$$(1) \qquad \gamma \, \mathrm{tg} \, \theta' = \sin \theta \, (\cos \theta + \beta E/pc)^{-1} = y \,,$$

where $\gamma = (1 - \beta^2)^{-1}$ is the Lorentz factor between the two reference frames. If the collision occurs at a height h from the ground the nucleon would land a distance $r = h \, \mathrm{tg} \, \theta'$ from the shower axis. From (1) $y = \gamma r/h$ is for a given γ/h a measure of the core spread. It is convenient to determine the distribution of cores in the laboratory system in terms of y.

The probability that a particle will be emitted between θ and $\theta + \mathrm{d}\theta$ in the F system is

$$(2) \qquad n(\theta) \, \mathrm{d}\theta = \tfrac{1}{2} N \sin \theta \, \mathrm{d}\theta \,,$$

where N is the total number of particles. The corresponding distribution in the laboratory system in terms of y is

$$(3) \qquad n(y) \, \mathrm{d}y = \tfrac{1}{2} N \sin \theta \, \mathrm{d}\theta/\mathrm{d}y \, \mathrm{d}y \,.$$

Putting $a = \beta E/pc$ and $z = \cos \theta$ in (1) we have

$$(4) \qquad y = (a + z)^{-1}(1 - z^2)^{\frac{1}{2}}$$

and

(5)
$$- \mathrm{d}z/\mathrm{d}y = y[y^2(a+z)+z]^{-1}(a+z)^2,$$

where

(6)
$$z = (1+y^2)^{-1}[-ay^2 \pm (1+y^2-a^2y^2)^{\frac{1}{2}}].$$

It is easily shown that for particles emitted forward from the F system $(\theta < \pi/2)$ only the $+$ sign in (6) is relevant so that

(7)
$$\mathrm{d}z/\mathrm{d}y = - y(1+y^2)^{-2}(1+y^2-a^2y^2)^{-\frac{1}{2}}[a + (1+y^2-a^2y^2)^{\frac{1}{2}}]^2$$

and the distribution of cores in the laboratory system from these forward particles is

(8)
$$n(y)\,\mathrm{d}y = \tfrac{1}{2}Ny(1+y^2)^{-2}(1+y^2-a^2y^2)^{-\frac{1}{2}}[a + (1+y^2-a^2y^2)^{\frac{1}{2}}]^2.$$

It is clear from (5) that for nucleons emitted forward in the F frame (z positive) $\mathrm{d}y/\mathrm{d}z$ is always negative. Those emitted at the large angles in the F frame land furthest from the shower axis. They extend out to a maximum given by (4) $y = 1/a$.

For particles emitted backwards in the F frame $\mathrm{d}z/\mathrm{d}y$ changes sign. As θ increases beyond $\pi/2$ the core will spread beyond $y = 1/a$ to a maximum at $y = (a^2 - 1)^{-\frac{1}{2}}$. Thereafter particles emitted at greater angles appear closer to the shower axis. Beyond the maximum spread we must choose the $-$ sign in (6).

While the spread of cores at the ground is increasing with increasing angle of emission in the F frame the core distribution is the same as (8)

(9)
$$n(y)\,\mathrm{d}y = \tfrac{1}{2}Ny(1+y^2)^{-2}(1+y^2-a^2y^2)^{-\frac{1}{2}}[a + (1+y^2-a^2y^2)^{\frac{1}{2}}]^2$$

and for greater angles of emission

(10)
$$n(y)\,\mathrm{d}y = \tfrac{1}{2}Ny(1+y^2)^{-2}(1+y^2-a^2y^2)^{-\frac{1}{2}}[a - (1+y^2-a^2y^2)^{-\frac{1}{2}}]^2.$$

The total distribution of cores at ground between $y = 0$ and $y = (a^2 - 1)^{-\frac{1}{2}}$ is thus

(11)
$$n_c(y) = Ny(1+y^2)^{-2}(1+y^2-a^2y^2)^{-\frac{1}{2}}[a^2 + 1 + y^2(1-a^2)].$$
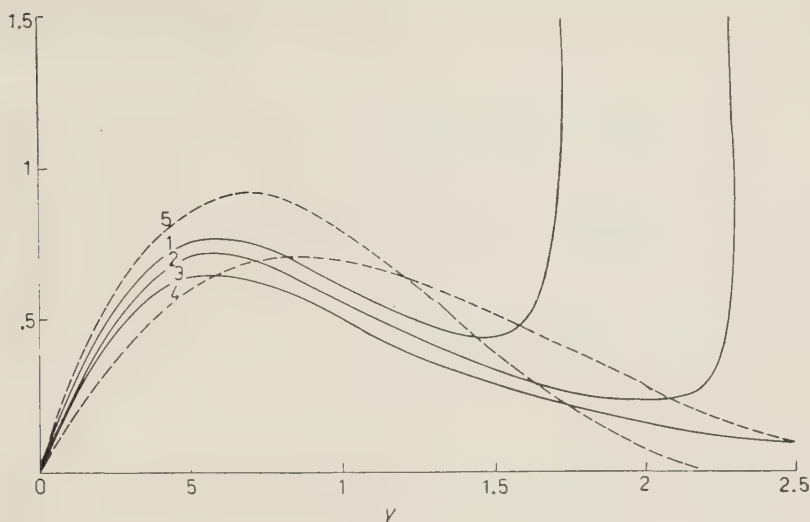
Fig. 2. – The probability that a shower core will fall at a given distance from the shower axis. Curves 1 and 2 are for the distribution (11) when the average nucleon kinetic energy is 1 and 1.5 GeV. Curve 3 is the distribution (12) and curves 4 and 5 the distributions (15) with different values of α.

The distribution (11) is plotted in Fig. 2 for two values of a corresponding to nucleons emitted in the F system with average energy 1 and 1.5 GeV and compared with the distribution for relativistic nucleons

$$(12) \qquad\qquad n_c(y) = 2Ny(1 + y^2)^{-2}$$

which is the special case of (11) when $a = 1$.

3. – The electron distribution.

The distribution of shower particles near the axis of showers begun by protons is not well known. It is therefore convenient to choose for the distribution about each core

$$(13) \qquad \varrho(r) = (2\pi)^{-1} N (\gamma_2 - \gamma_1)^{-1} \gamma_1 \gamma_2 r^{-1} \big(\exp\left[-\gamma_1 r\right] - \exp\left[-\gamma_2 r\right] \big) ,$$

where γ_1 and γ_2 are adjustable parameters and N is the total number of particles produced by each nucleon at the ground.

If the energy per nucleon is not too large in the F frame it is sufficient to suppose that all nucleons have the same energy at ground level.

The total distribution of electrons in a shower begun by a heavy primary is then

$$(14) \qquad \varrho(r') = \frac{N\gamma_1\gamma_2}{2\pi(\gamma_2 - \gamma_1)} \int n_c(r)\, d\boldsymbol{r}\, \varrho(\boldsymbol{r} - \boldsymbol{r}')\,,$$

where $n_c(r)$ is the density of cores normalized so that $\int n_c(\boldsymbol{r})\, d\boldsymbol{r} = 1$, N is the total number of electrons in the shower and both integrals are taken over the whole plane.

The integral (14) is difficult to evaluate if we use the core (11) directly. It is much more convenient to replace (11) with the function

$$(15) \qquad n_c(r) = -\alpha^2 \operatorname{kei}(\alpha r)\,,$$

where $\operatorname{kei}(\alpha r)$ is a function related to the Kelvin [11-13] functions.

Suitable choice of α can make the resulting electron distributions almost identical in the region of the core. The functions $\operatorname{kei}(x)$ has its first zero at $x = 3.9$ after which its magnitude rapidly decreases. Contributions to the integral (14) come almost wholly from $\alpha r \leqslant 3.9$ which we may take to be the extent of the core.

In terms of $y = \gamma r/h$ the spread of the core is determined by the kinetic energy of the nucleons in the F frame. Once this energy is fixed y max is a constant. In fitting the core with the function $-\alpha^2 \operatorname{kei}(\alpha r)$, α must be chosen so that

$$\alpha r_{\max} = \gamma^{-1}\, \alpha h y_{\max} = 3.9\,.$$

That is $\alpha h/\gamma$ must be kept constant.

In Fig. 2 typical fits to the core structure are shown as a function of y. Using (15), (14) becomes

$$(16) \quad \varrho(r)' = -\frac{N\gamma_1\gamma_2\alpha^2}{2\pi(\gamma_2 - \gamma_1)} \int d\boldsymbol{r}\, \operatorname{kei}(\alpha r) \frac{\exp\left[-\gamma_1|\boldsymbol{r} - \boldsymbol{r}'|\right] - \exp\left[-\gamma_2|\boldsymbol{r} - \boldsymbol{r}'|\right]}{|\boldsymbol{r} - \boldsymbol{r}'|}\,.$$

where N is the total number of shower particles.

Making use of the Fourier transform of (13), (16) becomes

$$(17) \qquad \varrho(r') = -\frac{N\gamma_1\gamma_2(\gamma_2 + \gamma_1)\alpha^2}{4\pi^3} \int d\boldsymbol{r}\, \operatorname{kei}(\alpha r) \int \frac{d\boldsymbol{k}\, \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}')\right]}{(k^2 + \gamma_1^2)(k^2 + \gamma_2^2)}$$

[11] C. N. WATSON: *Theory of Bessel Functions*, (Cambridge, 1952) p. 81.
[12] N. W. McLACHLAN: *Bessel Functions for Engineers* (Oxford, 1934), pp. 119, 168.
[13] H. B. DWIGHT: *Tables of Integrals and other Mathematical Data* (London, 1947), p. 240.

and using the zero order Bessel transform of (15)

$$
(18) \qquad \varrho(r') = \frac{2N\gamma_1\gamma_2(\gamma_1 + \gamma_2)\alpha^4}{\pi r'} \int_0^\infty \frac{k\,dk\,\sin kr'}{(k^2 + \gamma_1^2)(k^2 + \gamma_2^2)(k^4 + \alpha^4)}.
$$

The form (18) can be integrated immediately to give for the electron distribution from a heavy primary

$$
(19) \qquad \varrho(r') = N\gamma_1\gamma_2(\gamma_1 + \gamma_2)\alpha^4(2\pi r')^{-1}\cdot
$$

$$
\cdot(\gamma_1^2 - \gamma_2^2)^{-1}\{(\alpha^4 + \gamma_1^4)^{-1}\exp\left[-\gamma_1 r'\right] - (\alpha^4 - \gamma_2^4)^{-1}\exp\left[-\gamma_2 r'\right]\} +
$$

$$
+ \alpha^{-2}(\gamma_1^4 + \alpha^4)^{-1}(\gamma_2^4 + \alpha^4)^{-1}\exp\left[\frac{-\alpha r'}{\sqrt{2}}\right]\left[(\gamma_1^2\gamma_2^2 - \alpha^4)\sin\frac{\alpha r'}{\sqrt{2}} - (\gamma_1^2 + \gamma_2^2)\alpha^2\cos\omega\frac{\alpha r'}{\sqrt{2}}\right].
$$

The distribution (19) is plotted in Fig. 3 for different values of (α, γ_2). γ_2 controls the steepness of the electron distribution about separate cores and
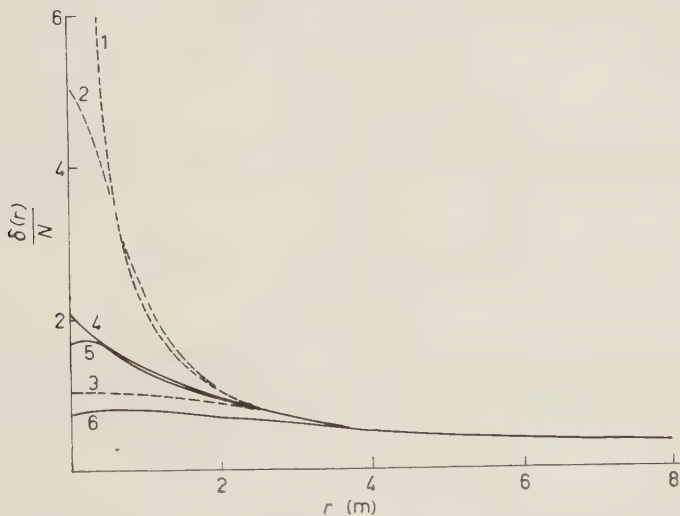
Fig. 3. – Curves 2 and 3 represent the function (19) which gives the distribution about the shower axis of electrons in a shower begun by a heavy primary, for different spreads of the core. The parameters used were $\gamma_1 = .013$, $\gamma_2 = 10$, $\alpha = 3.5$ and $\gamma_1 = .013$, $\gamma_2 = 10$, $\alpha = .7$. Curves 5 and 6 correspond to 2 and 3 except that for them the electron density about single cores is much less steep near the cores. For curves 5 and 6 the parameter γ_2 was taken to be 1. Curves 1 and 4 are the distribution (13) of electrons about single cores used in calculating curves 2, 3, and 5, 6. For curve 1 γ_2 was chosen to be 10 and for curve 4 $\gamma_2 = 1$.

α the spread of the cores. It is clear from Fig. 3 that as α gets large the region of the shower for which the density is sensitive to changes in α grows very small. It is important that for the majority of showers which the M-unit detects, α should not be too large.

4. – Counting rates.

It remains to determine what effect tidal oscillations of the interaction level will have on the M-unit counting rate.

If a shower falls with its axis at r from the centre of the M-unit the probability that it will trigger the i-th counter is

$$P_i = 1 - \exp\left[-x_i\right],$$

where $x_i = A\varrho(r_i)$ and A is the area of the counter. The probability that the M-unit will trigger is

$$(20) \qquad P(r, \theta) = \prod_{i=1}^{3} \left(1 - \exp[-x_i]\right).$$

The probability that a shower of size N will fall at the distance r is

$$N^{-\gamma} r \, \mathrm{d}r \, \mathrm{d}N \, \mathrm{d}\theta \,,$$

where γ is the shower size exponent, so that the total probability of triggering the M-unit is

$$(21) \qquad P_T = \int \mathrm{d}N \, \mathrm{d}\theta \, \mathrm{d}r \, r \, N^{-\gamma} \left[\prod_{i=1}^{3} (1 - \exp\left[-x_i\right]) \right].$$

The relative triggering probability for showers interacting at various levels has been determined by integrating (21) numerically using the electron distribution (19) for various assumptions about the nature of the primary event, as a function of the tidal amplitude, and for different distributions about single cores.

The two gross features to which the relative change of P_T with tidal motion is sensitive are changes in γ and elasticity. However, in all cases the observed change in rate appears to indicate very large tidal amplitude of the level of interaction.

5. – Discussion of results.

A change in the height of the first interaction changes the electron density mainly in the area covered by the nucleon core.

As the primary energy increases it is to be expected that the Lorentz factor of the F frame with respect to the ground will increase and the core will shrink.

It may be seen from Fig. 3 that when $\alpha = 3.5$ the region over which the electron density changes appreciably is very small.

With increasing shower size, the electron density outside the core becomes sufficient to trigger the M-unit. The density inside becomes so large that the M-unit is saturated. It is clear that for showers greater than a certain size the counting rate is no longer sensitive to the height of the first interaction From Figs. 4, 5, 6, this size can be seen to be about $2 \cdot 10^6$ particles.
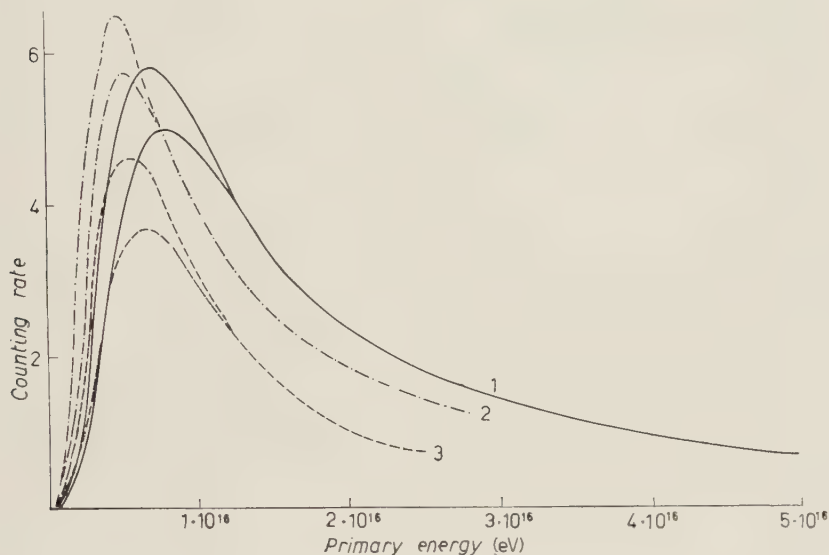
Fig. 4. – Counting rates for showers as a function of the primary energy. Each pair of curves gives the maximum and minimum counting rates for a tidal amplitude of 7 km in the interaction level. The core in all cases is assumed to shrink directly with the primary energy. Curves 1 and 2 differ in the spread of core assumed for a given energy. The parameters for them are $\gamma_1 = .013$, $\gamma_2 = 15$, $\alpha = .08 N/h$, $\gamma = 1.5$ and $\gamma_1 = .013$, $\gamma_2 = 15$, $\alpha = 2N/h$, $\gamma = 1.5$. Curves 1 and 3 differ in the frequency assumed for primaries of different energies. For curve 3 the value chosen for γ was $\gamma = 2$.

Saturation of the M-unit occurs whenever the density is high. The probability that the M-unit will be triggered by a shower whose axis falls a distance r from the centre of the unit is given by

$$P(r, \theta) = \prod_{i=1}^{3} \left(1 - \exp\left[-x_i\right]\right),$$

where $x_i = A\varrho(r_i)$ and A is the counter area. When $\varrho(r_i) > 10^3$ particles/m² the M-unit is saturated.
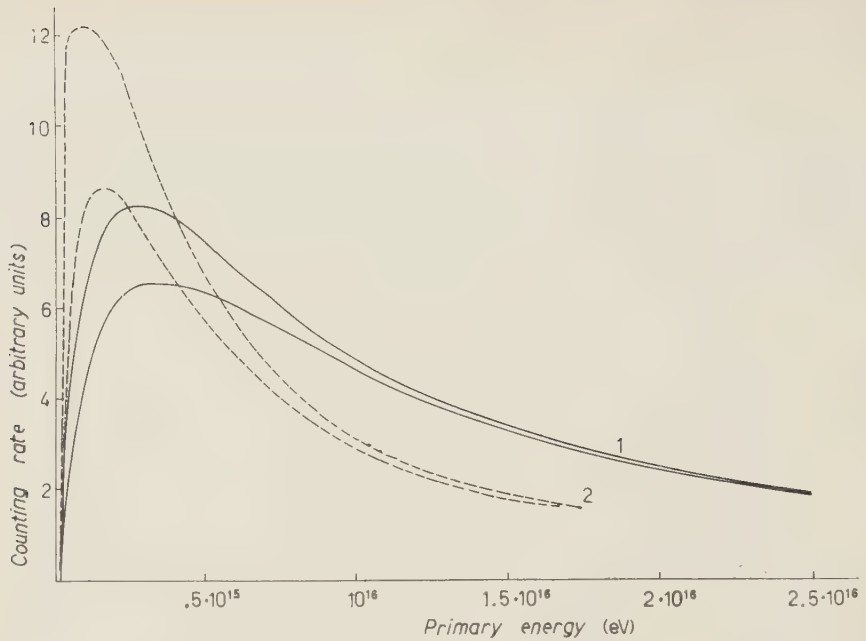
Fig. 5. – Counting rates for showers as a function of the primary energy for constant core size. The parameters used in the expression (21) were for curve 1, $\gamma_1 = .013$, $\gamma_2 = 15$, $\gamma = 1.5$ and $x = 1.2$; and for curve 2, the same except that the exponent γ in the size frequency spectrum was taken to be 2.
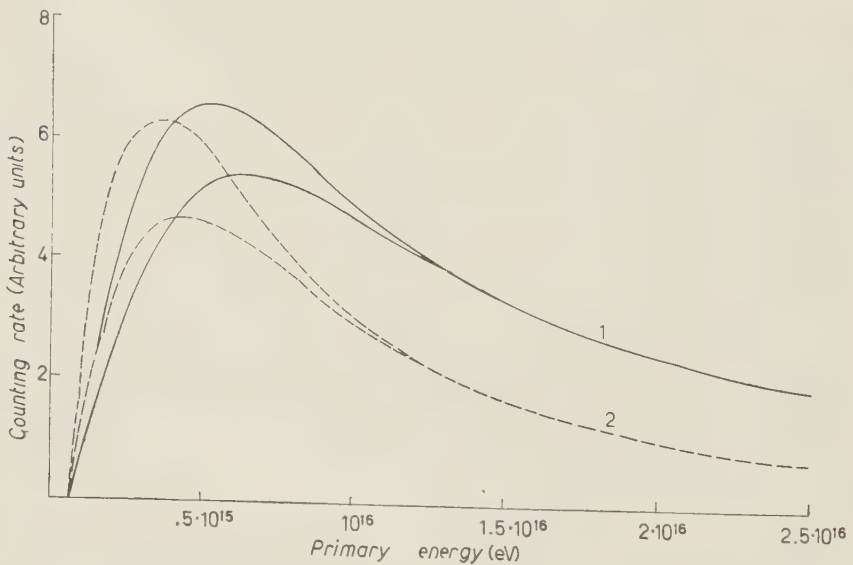

Fig. 6. – Counting rates for showers as a function of the primary energy for a core which shrinks as $E^{-\frac{1}{4}}$. Curves (1) and (2) correspond to those of Fig. 5 except that parameters α, controlling the core spread was taken as $\alpha = 1.55 \cdot 10^7 \, E^{-\frac{1}{4}}$.

The region of the shower which can make the total counting rate sensitive to the height of the first interaction is where the density is sufficiently high to often trigger the unit but not high enough to saturate it.

For small showers this region lies wholly within the core, for larger showers it extends beyond. The counting rate is made more sensitive to the height of the first interaction when the rôle of small showers is emphasized or when the core is extended for larger showers. In Fig. 4 the maximum and minimum counting rates are plotted for a tidal amplitude of 7 km in the mean inter-action level, for three different assumptions. (Each pair of curves is arbitrarily normalized.) In each case the core was assumed to shrink directly as the primary energy increased.

The observations of DMITRIEV et al. ([7]), suggest that the spread of the core is not so dependent on the primary energy. These authors found that the high energy nuclear active particles in showers of size 10^5 to be spread over an area not very different from that for showers of size 10^6. There is evidence that m^* increases with the primary energy. PETERS has suggested also that the mean size of the primaries increases rapidly with shower size.

In Figs. 5 and 6 the counting rates are plotted for constant core spread and for a core whose radius varies as $E^{-\frac{1}{2}}$. In both cases the counting rate is much more sensitive to the height of the first interaction than for Fig. 4.

An increase in the exponent γ in the primary energy spectrum increases the contribution of small showers to the total counting rate. According to CLARK ([14]) γ may greater than 2. The effect on the counting of a change in the level of the first interaction depends very strongly on the values assumed for γ.

Changing the steepness of the electron distributions about single cores did not have a marked effect on the relative difference in counting rate. From Fig. 3 it may be seen that shallow distributions increase the sensitive area for large showers by decreasing the electron density near to the shower axis. However, this decrease sharply cuts the counting rate for smaller showers.

Table I is a summary of the amplitudes found for the variation in counting rate and the assumptions made.

Figs. 5 and 6 correspond best with experimental observations near to the shower core. However, so little is known about the primary spectrum, the cross-sections at such high energies, and the actual densities in and above the ozone layer that it is difficult to assign a definite value to the tidal ampli-tude at the height of the first interaction. *When more is known of the structure of showers near to the core it seems likely that suitable geiger arrays will provide the best method of observing movements of the upper atmosphere. Our analysis*

([14]) G. CLARK, J, EARL, W. KRAUSHAAR, J. LINSLEY, B. ROSSI and F. SCHERB: *Nature* (*Lond.*), **180**, 353 and 406 (1957).

TABLE I.

Curve No.	Figure No.	Core spread for 10^5 shower	Dependence of core spread on energy	Value of γ	Ampl. of change in counting rate
1	4	20 metres	$\propto E^{-1}$	1.5	4%
2	4	8 metres	$\propto E^{-1}$	1.5	3%
3	4	20 metres	$\propto E^{-1}$	2	9%
1	5	3.3 metres	const	1.5	7%
2	5	3.3 metres	const	2.0	15%
1	6	8 metres	$E^{-\frac{1}{2}}$	1.5	6%
2	6	8 metres	$E^{-\frac{1}{2}}$	2.0	13%

even at present shows that the large solar variations seen by McCusker imply a tidal amplitude of the 40 km level of between three and ten kilometres.

There have been few attempts to observe tidal motions in the ozone layer, the observers being mainly concerned with prevailing seasonal winds. It is well known that large tidal motions occur in the E layer at 80 km where tidal winds with amplitudes of up to 100 m/s have been observed with large vertical components.

KENNEDY [15] using the anomalous propagation of sound waves found in the 40 km region, a ten degree change in temperature throughout a day accompanied by diurnal winds with amplitude of 30 m/s.

WEEKES and WILKES [16] discussed the production of atmospheric tides in this region in terms of the trapping energy by the low temperature « barrier » which is known to exist at about 80 km. They have extended the resonance theory of PECKERIS [17] and TAYLOR [18] and have found that the sharpness of the resonance is strongly dependent on the thickness of the cold layer which determines the efficiency with which the energy is trapped. If this trapping were as strong as in the second model they discuss the fluctuations in density would be ample to explain McCusker's result. The phase of the variation is also in keeping with the resonance theory. PECKERIS predicts a change in the phase of the twelve hour wave above 30 km. The maximum in McCusker's counting rate occurs with the maximum pressure at ground,

[15] W. B. KENNEDY, L. BROGAN and N. J. SABLE: *Journ. Met.*, **12**, 519 (1955).
[16] K. WEEKES and M. V. WILKES: *Proc. Roy. Soc.*, A **192**, 80 (1947).
[17] C. L. PEKERIS: *Proc. Roy. Soc.*, A **158**, 650 (1937).
[18] G. I. TAYLOR: *Proc. Roy. Soc.*, A **156**, 318 (1936).

that is, out of phase with the motion of the interaction level. On our model increasing the height of the interaction decreases the density near to the core and the counting rate.

<p style="text-align:center">* * *</p>

RIASSUNTO (*)

Si è fatto un tentativo di spiegare le variazioni solari nel conteggio degli sciami estesi dell'aria, riferito da McCUSKER, in base ad uno spostamento di marea nell'altezza della prima interazione del primario. È possibile ottenere le grandissime ampiezze trovate da McCUSKER con una ampiezza del moto di mare dello strato di ozono tra i 3 e i 10 km, e purchè quasi tutti i primari di alta energia siano presenti.

(*) *Traduzione a cura della Redazione.*

Hyperfragment Production in K⁻-Meson Absorptions at Rest in Nuclear Emulsions. Non-Mesic Decays.

J. Sacton (*)

Université Libre - Bruxelles

Summary. — The results of the analysis of 78 non-mesic hyperfragments including double centered stars produced in K⁻-meson absorptions at rest are given. The Z-spectrum of the hyperfragments is studied. 12 decays have been uniquely identified and in 6 other cases the charge of the hyperfragment has been unambigously determined. For the remaining events only limits could be put to the charge of the hyperfragment. Combining the charge distribution established here with that obtained for the mesic hyperfragments in a previous investigation, one can obtain the non-mesic to mesic decay ratio as a function of Z.

1. – Introduction.

In an earlier report [1], hereafter referred to as paper I, we have described 26 hyperfragments decaying mesically produced in K⁻-meson absorptions at rest in nuclear emulsions. The purpose of the present paper is to report our results about the analysis of some 56 non-mesic hyperfragments found during the same experiment. As « non-mesic », we consider all hyperfragments decaying without the emission of a negative π-meson, *i.e.*, those for which the decay proceeds with real neutral pion emission:

$$(1) \qquad \Lambda^0 \to \pi^0 + n + 42 \text{ MeV}$$

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[1] J. Sacton: *Nuovo Cimento*, **15**, 110 (1960).

or non-mesically, through one of the following stimulated processes introduced a few years ago by W. CHESTON and H. PRIMAKOFF [2]:

$$(2) \qquad \Lambda^0 + n \to n + \pi^0 + n \to n + n$$

$$(3) \qquad \Lambda^0 + p \to p + \pi^- + p \to p + n \left.\right\} + 176 \text{ MeV}.$$

$$(4) \qquad \Lambda^0 + p \to n + \pi^0 + p \to n + p$$

2. – Experimental procedure.

2˙1. *Exposure, scanning and measurements.* – 56 events which exhibit the characteristics of a non-mesic hyperfragment decay star and 27 double centered stars with connecting track shorter than 2 μm have been located during the scanning of two emulsion stacks exposed at the Berkeley Bevatron (stack K_1: 300 Ilford G-5 emulsion sheets; stack K_2: 300 Ilford K-5 emulsion sheets) The scanning and measurement methods are essentially the same as in paper I.

2˙2. *Hyperfragment identification.* – The experimental biases affecting the detection of the non-mesic hyperfragments decaying into two or more charged particles can in many cases be minimized by careful observation—except perhaps in the case of cryptofragments. However it is not possible to identify non-mesic decay modes leading to the emission of a single prong or nuclear recoil. The non-mesic decay of hyperhydrogen, for instance, cannot be identified. Moreover, in addition to these biases there exist also the possibility of confusing short range non-mesic hyperfragments with other phenomena. The results of a systematic study of different background phenomena simulating hyperfragment decays are as follows:

a) an estimate of the random coincidence of a K⁻-meson ending and a background radioactive or small cosmic star is about 10^{-6};

b) the probability that a particle emitted in a K⁻-meson star interacts in flight after a range shorter than 5 μm is less than 10^{-5};

c) to our knowledge only one example of a K⁻-meson interacting nearly at rest ($E < 5$ MeV) and re-emerging from the interaction has been reported until now [*];

d) at most 2 events can be due to the absorption at rest of π^--mesons emitted with very low energy in the K⁻-meson star [3];

[2] W. CHESTON and H. PRIMAKOFF: *Phys. Rev.*, **92**, 1537 (1953).
[*] Private communication from Dr. E. H. S. BURHOP.
[3] M. SHAFI and D. J. PROWSE: *Proc. of the Padua-Venice Conference* (1957).

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e) a systematic comparison of the characteristics (*) of well identified Σ^--hyperon absorption stars and *definite* non-mesic hyperfragment decay stars (see definition of « definite » non-mesic hyperfragments below) leads us to estimate the contamination of short range Σ^--hyperons to our sample of events to be less than 10%.

In conclusion, the only relevant background contribution is due to the short range Σ^--hyperons. We are thus led to consider two classes of events:

a) the *definite* non-mesic hyperfragments and

b) the other events.

The so-called definite non-mesic hyperfragments satisfy at least one of the following criteria:

a) the energy balance of the K^--meson star excludes the possibility of a Σ^--hyperon being emitted;

b) the K^--meson star contains a π^--meson (the existence of K^--meson stars leading to the emission of a π^--meson together with a Σ^--hyperon has not yet been firmly established);

c) measurements performed on the hyperfragment track exclude any other interpretation.

19 such events have been identified during this work.

2˙3. *Analysis.* – The detailed kinematic analysis of the stimulated decays (reaction 2 to 4) is made difficult by the large amount of energy release ($Q \approx 180$ MeV) which presumably tend to favour the emission of more than one neutron during the decay process thus excluding in many cases the possibility of a unique identification. On the other hand, assuming the same processes for the neutral mesic decays as for the negative ones, the most frequent neutral decay schemes would be of type:

$$(5) \qquad\qquad {}_\Lambda X_z^A \rightarrow \pi^0 + n + X_z^{A-1}$$

involving also two neutral particles.

Owing to this complexity, as has been usual in previous work we consider among all kinematically possible solutions, only those involving a minimum number of neutral particles.

——— ———   .

(*) The criteria on which our comparison is based are essentially:
   1) the size of the star;
   2) the yield of energetic protons ($E_p > 30$ MeV), and
   3) the presence of Auger electrons.

In order to simplify further, we assume that:

a) all the decays take place at rest (in fact, no clear example of decays in flight have been detected in this work);

b) the short range nuclear splinters emitted in many cases are at most of charge 6.

The kinematic analysis of the decays has been performed by use of the Gamma 3 B Bull electronic computer. The code was based on one designed by E. M. SILVERSTEIN ([4]). 47 events with at most four prongs were completely analyzed. In 3 of the discarded hyperfragments, one of the decay products interacted in flight or left the stack after a range too short to allow accurate energy determination by ionization or scattering measurements. In another 3 events, the geometry of the decay star precluded the carrying out of complete angular measurements so that it was not possible to calculate any momentum balance. The remaining 3 stars consisted of five or more prongs and were not analyzed owing to the unfructuability of designing special codes for only one or two events.

3. – Experimental results.

The general characteristics of the 56 non-mesic decay stars are given in Table I; the events indicated by an * satisfy the above criteria for classification as definite hyperfragments. Column (2) contains the proposed identifications. In column (5) we give the number and nature of the neutral particles involved in the decay.

Only one event (Bx 2061) could be interpreted as a decay into charged particles only. The decay star contains 4 prongs: *a*) which is due to an energetic deuteron identified by ionization measurements; *b*) which is $2\,230$ µm long and is due to a singly charged particle of undetermined mass; *c*) which is caused by a doubly charged splinter of 39 µm range and *d*) which is due to a nuclear recoil. The different processes satisfying the kinematics of the decay are summarized in Table II. It must be emphasized that this event is also consistent with other interpretations involving one or more neutrons which are not detailed here.

The 46 other completely analyzed hyperfragments have been divided into two classes: *a*) those which are consistent with schemes involving only one neutral particle and *b*) those of which the decay involves more than one neutral particle. We shall consider now these 2 types of event separately.

([4]) E. M. SILVERSTEIN: *Suppl. Nuovo Cimento*, **1**, 41 (1959).

TABLE I. – *Characteristics of the hyperfragment decay stars.*

No. event	Identity	Track	Range (μm)	No. and nature of neutral particles
(1)	(2)	(3)	(4)	(5)
Gö 40	He$_\Lambda$	pdt pdt	10 519 3 320	2n
Gö 91	Be$_\Lambda$ or B$_\Lambda$	pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	162 141 86	1n
Gö 128	^{9}Be$_\Lambda$	d ^{6}Li	8 304 5.6	1n
Mi 154	Li$_\Lambda$ or Be$_\Lambda$	p pdt pdt ^{3}He ^{4}He	33 211 2 573 189	2n
Sa 10	$Z \geqslant 3$	pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He recoil	178 110 2.0	2n
Sa 33	$Z \geqslant 3$	pdt pdt ^{3}He ^{4}He recoil	28 800 81 7	2n
Sa 106	Li$_\Lambda$ or Be$_\Lambda$	pdt pdt pdt ^{3}He ^{4}He	26 040 67 57	1n
Bx 4	He$_\Lambda$ or Li$_\Lambda$	pdt pdt ^{3}He ^{4}He	619 62	2n
Bx 133	^{8}Be$_\Lambda$	d p ^{4}He	5 952 78.1 19.4	1n
Bx 147	Be$_\Lambda$ or B$_\Lambda$	p pdt pdt recoil	9 310 2 713 503 8.0	1n
Bx 159	He$_\Lambda$, Li$_\Lambda$ or Be$_\Lambda$	pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	359.8 40.0	2n
Bx 2000	$Z \geqslant 4$	p pdt recoil recoil	> 13 000 434 19.9 10.8	?

TABLE I (*continued*).

No. event	Identity	Track	Range (μm)	No. and nature of neutral particles
(1)	(2)	(3)	(4)	(5)
Bx 2021	He_Λ or Li_Λ	pdt pdt ^{3}He ^{4}He	22 304 340	2n
Bx 2028	$Z \geqslant 3$	pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He recoil	512.1 474.2 ~ 1.0	2n
Bx 2061	see text	d pdt pdt ^{3}He ^{4}He recoil	11 127 2 230 38.5 3.6	—
Bx 2085	$^7He_\Lambda$	t t	673 529	1n
Bx 2100	$Z \geqslant 5$	p pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	21 317 388.5 234.6 160.7 28.6	?
Bx 2122	$Z \geqslant 3$	pdt pdt recoil	8 750 299 2.9	2n
Bx 2146	$^{12}C_\Lambda$	p ^{4}He ^{6}Li	9 235 228.9 5.1	1n
Bx 2229	B_Λ, C_Λ or N_Λ	d ^{3}He ^{4}He pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	14 333 66.7 29.2 17.5	1n
Bx 2261	$Z \geqslant 3$	pdt pdt ^{3}He ^{4}He recoil	5 104 878.7 ~ 2	?
Bx 2303	He_Λ, Li_Λ or Be_Λ	pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	735 324	2n
Bx 3072	$^5He_\Lambda$	pd pd	7 023 3 665	2n

TABLE I (*continued*).

No. event	Identity	Track	Range (μm)	No. and nature of neutral particles
(1)	(2)	(3)	(4)	(5)
Bx 4033	$^4He_\Lambda$	p d	2 238 1 580.5	1n
Bx 4087	$^{10}B_\Lambda$ or $^{11}B_\Lambda$	pdt $^3He\ ^4He$ $^3He\ ^4He$	146.5 76.3 59.1	1n
Bx 4116	He_Λ or Li_Λ	pdt $^3He\ ^4He$ pdt	295.2 > 100	?
Bx 4124	He_Λ or Li_Λ	pdt pdt $^3He\ ^4He$	8 609 181.1	?
Bx 4126	$Z \geqslant 6$	pdt $^3He\ ^4He$ pdt $^3He\ ^4He$ pdt $^3He\ ^4He$ pdt $^3He\ ^4He$ pdt $^3He\ ^4He$ recoil	446.8 164.0 121.7 93.2 60.9 3.0	?
Bx 5003	$^6Li_\Lambda$	p ^4He	16 720 848	1n
Bx 5004	$^5He_\Lambda$	t p	1 303 1 071.8	1n
Bx 5005	Be_Λ or B_Λ	pdt $^3He\ ^4He$ pdt $^3He\ ^4He$ pdt $^3He\ ^4He$	446.0 244.2 133.5	1n
Bx 5006	B_Λ or C_Λ	pdt pdt $^3He\ ^4He$ pdt $^3He\ ^4He$	233.0 199.7 123.3 71.0	1n
Bx 5007	$^9Be_\Lambda$	d ^6Li	8 636 7.5	1n
Bx 5007b	$Z \geqslant 2$	pdt $^3He\ ^4He$ recoil	61.0 ~ 1	?
Bx 5008	Li_Λ, Be_Λ or B_Λ	p pdt $^3He\ ^4He$ pdt $^3He\ ^4He$	22 340 108.1 22.9	2n

TABLE I (continued).

No. event	Identity	Track	Range (μm)	No. and nature of neutral particles
(1)	(2)	(3)	(4)	(5)
Bx 5011	Li$_\Lambda$ or Be$_\Lambda$	pdt pdt pdt ^{3}He ^{4}He	1 044 174 67	1n
Bx 5013	^{8}Be$_\Lambda$	p d ^{4}He	12 827 4 961 80.1	1n
Bx 5014	$Z \geqslant 2$	pdt pdt ^{3}He ^{4}He	2 491 14.9	2n
Bx 5015	$Z \geqslant 2$	pdt recoil	15 613 4.56	2n
Bx 5019	$Z \geqslant 5$	pdt pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He recoil	1 690.3 312.6 170.9 45.7 2.1	?
Bx 5021	Li$_\Lambda$	p pdt pdt	12 081 1 717 1 331	2n
Bx 5022	B$_\Lambda$ or C$_\Lambda$	^{3}He ^{4}He pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	424.6 27.3 10.0	1n
Bx 5023	^{9}Be$_\Lambda$	p ^{7}Li	8 579 3.0	1n
Bx 5024	B$_\Lambda$ or C$_\Lambda$	p pdt recoil	2 777 231.7 4.7	1n
Bx 5025	$Z \geqslant 3$	pdt pdt ^{3}He ^{4}He recoil	678.5 119.9 3.0	2n
Bx 5028	He$_\Lambda$ or Li$_\Lambda$	p recoil	14 116 6.7	1n
Bx 5030	^{4}He$_\Lambda$	d d	61.9 6.9	π^0-meson

TABLE I (*continued*).

No. event (1)	Identity (2)	Track (3)	Range (μm) (4)	No. and nature of neutral particles (5)
Bx 5031	He$_\Lambda$ or Li$_\Lambda$	pdt pdt ^{3}He ^{4}He	> 7 635 294.2	?
Bx 5041	^{11}Be$_\Lambda$	p ^{8}Li	5 101 8.1	2n
Bx 5046	$Z \geqslant 4$	p pdt pdt ^{3}He ^{4}He recoil	21 393 2 102 203 5.7	2n
Bx 5047	B$_\Lambda$ or C$_\Lambda$	p pd pdt ^{3}He ^{4}He ^{3}He ^{4}He	41 033 136 96.6 48.6	1n
Bx 5048	Li$_\Lambda$, Be$_\Lambda$ or B$_\Lambda$	pdt pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	7 169 383.5 56.2	2n
Bx 5050	$Z \geqslant 2$	pdt ^{3}He ^{4}He recoil	141.8 1.2	2n
Bx 5051	$Z \geqslant 2$	pdt ^{3}He ^{4}He recoil	150.6 2.4	π^0-meson or 2n
Bx 5052	^{7}Li$_\Lambda$ or ^{8}Li$_\Lambda$	t d recoil	1 284.5 261.7 ~1.0	1n
Bx 5053	^{7}Li$_\Lambda$	pdt ^{3}He ^{4}He pdt ^{3}He ^{4}He	352.4 317	1n

TABLE II. – *Event* Bx 2061.

Decay scheme	B_Λ (MeV)	ΔB_Λ (*) (MeV)	Q (MeV)	P_r (**) (MeV/c)
^{13}C$_\Lambda \rightarrow$ d + t + d + ^{6}Li	13.5	3.0	114.8	39
^{14}C$_\Lambda \rightarrow$ d + t + d + ^{7}Li	15.8	3.0	114.8	30
^{14}N$_\Lambda \rightarrow$ d + t + d + ^{7}Be	17.1	3.0	114.9	29

(*) ΔB_Λ = error on B_Λ.
(**) P_r = residual momentum.

3˙1. *Decays involving only one neutral particle.* – 25 hyperfragments belong to this class; 11 of them are uniquely identified *i.e.*, only one interpretation leads to an acceptable value for the Λ^0-hyperon binding energy. The decay schemes of these events are given in Table III together with the correspond-

TABLE III. – *List of the uniquely identified hyperfragments of which the decay involves one neutral particle.*

No. event	Decay scheme	B_Λ (MeV)	Q (*) (MeV)	E_n (**) (MeV)
Bx 4033	$^4He_\Lambda \to p+d+n$	7.2	46.6	116.2
Bx 5004	$^5He_\Lambda \to t+p+n$	3.0	40.3	112.5
Bx 2085	$^7He_\Lambda \to t+t+n$	3.8	32.3	127.1
Bx 5003	$^6Li_\Lambda \to p+{}^4He+n$	7.3	120.9	49.5
Bx 133	$^8Be_\Lambda \to d+p+{}^4He+n$	4.9	60.1	103.5
Bx 5013	$^8Be_\Lambda \to p+d+{}^4He+n$	7.3	119.4	41.7
Bx 5007	$^9Be_\Lambda \to d+{}^6Li+n$	8.2	68.9	76.1
Gö 128	$^9Be_\Lambda \to d+{}^6Li+n$	6.6	66.7	80.0
Bx 5023	$^9Be_\Lambda \to p+{}^7Li+n$	6.8	49.9	101.6
Bx 2146	$^{12}C_\Lambda \to p+{}^4He+{}^6Li+n$	11.8	76.1	74.6
Bx 5030	$^4He_\Lambda \to d+d+\pi^0$	— 2.3	68.8	31.0

(*) Q = total kinetic energy of the charged products.
(**) E_n = kinetic energy of the neutral particle.

ing B_Λ- and Q-values. The following remarks can be drawn up about the results contained in this table:

a) event Bx 2085 is interpreted as an hyperhelium of mass 7 decaying into two tritons and a neutron and is characterized by a binding energy for the Λ^0-hyperon equal to (3.8 ± 3.0) MeV. Only 4 examples of mass 7 hyperhelium have been identified until now. Two of them decaying through the same process as above were found by E. M. SILVERSTEIN [4] but no binding energy has been computed by the author. Another one reported by R. G. AMMAR and coll. [5] decays mesically through the following scheme:

$$^7He_\Lambda \to \pi^- + {}^3H + {}^4He .$$
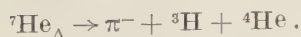
The binding energy of the Λ^0-hyperon inferred from the kinetic analysis of the decay star was found to be (3.0 ± 0.7) MeV (*). The last event has been

(⁵) R. G. AMMAR, R. LEVI SETTI, W. E. SLATER, S. LIMENTANI, P. E. SCHLEIN and P. H. STEINBERG: *Nuovo Cimento*, **15**, 181 (1960).
(*) Calculated with $Q_{\Lambda^0} = (37.58 \pm 0.15)$ MeV.

reported by us in paper I. This hyperfragment was not uniquely identified; however one of the proposed decay schemes

$$^7\text{He}_\Lambda \to {}^7\text{Li} + \pi^-$$
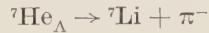
led to a B_Λ-value of (5.42 ∓ 1.50) MeV. If the recoiling nucleus is emitted in its 0.48 MeV excited state, the binding energy becomes around 5 MeV.

The mean binding energy deduced from these 3 events (~ 4 MeV) is rather low compared with the binding energy (~ 6 MeV) given for $^7\text{Li}_\Lambda$ by R. G. AMMAR and coll. ([5]). This result seems to confirm the suggestion proposed by these authors according to which $^7\text{He}_\Lambda$ and $^7\text{Li}_\Lambda$ do not belong to the same isotopic spin multiplet;

b) event Bx 5023 illustrates a new decay mode of $^9\text{Be}_\Lambda$:

$$^9\text{Be}_\Lambda \to p + {}^7\text{Li} + n .$$
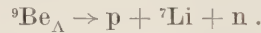
The measured binding energy is 6.8 MeV. From the three examples of $^9\text{Be}_\Lambda$ mentioned in Table III, we deduce a mean value of the Λ^0-hyperon binding energy equal to 7.2 MeV in good agreement with that one given by R. G. AMMAR and coll. ([5]) *i.e.* (6.4 ± 0.4) MeV.

c) the hypercarbon Bx 2146 detected in the course of this work is to our knowledge the heaviest hyperfragment uniquely identified until now:

$$^{12}\text{C}_\Lambda \to p + {}^4\text{He} + {}^6\text{Li} + n ;$$

*d*) event Bx 5003 adds itself to the enigmatic category of the possible mass 6 hyperfragments;

e) in only one case (Bx 5030) is the neutral particle involved in the decay a π^0-meson.

TABLE IV. – *Uniquely charge determined hyperfragments of which the decay involves one neutral particle.*

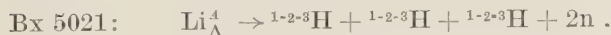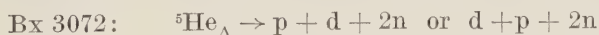| No. event	Decay scheme	B_Λ (MeV)	Q (MeV)	E_n (MeV)
Bx 5052	$^7\text{Li}_\Lambda \to t + d + p + n$	7.9	31.5	115.0
	$^8\text{Li}_\Lambda \to t + d + d + n$	6.8	31.6	111.0
Bx 5053	$^7\text{Li}_\Lambda \to d + {}^4\text{He} + n$	3.3	38.3	132.4
	$^7\text{Li}_\Lambda \to t + {}^3\text{He} + n$	9.1	37.0	113.7
	$^7\text{Li}_\Lambda \to {}^3\text{He} + t + n$	5.7	37.9	116.2
Bx 4087	$^{10}\text{B}_\Lambda \to p + {}^4\text{He} + {}^4\text{He} + n$	15.7	26.3	133.8
	$^{10}\text{B}_\Lambda \to d + {}^3\text{He} + {}^4\text{He} + n$	12.8	26.5	118.1
	$^{10}\text{B}_\Lambda \to t + {}^3\text{He} + {}^3\text{He} + n$	3.8	26.4	112.9
	$^{11}\text{B}_\Lambda \to d + {}^4\text{He} + {}^4\text{He} + n$	3.8	26.7	127.0

Data on another 3 hyperfragments, for which only the charge could be uniquely identified, are given in Table IV. For the remaining 11 events, the analysis provides only limits for the hyperfragment charge (see Table I, column 2).

3'2. *Decays involving two or more neutral particles.* – The information that can be gained from this category of hyperfragments is rather poor. Out of a total of 21 decays involving the emission of two or more neutral particles only one event (Bx 5041) could be considered as uniquely identified. Its decay star consists of two prongs: the first one is $5100 \mu m$ long and is due to a singly charged particle and the second one ends after a range of $8.1 \mu m$ in the characteristic hammer shape. The only possible decay scheme is:

$$^{11}\text{Be}_\Lambda \rightarrow {}^8\text{Li} + \text{p} + \text{n} + \text{n}$$
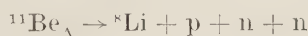
leading to binding energy values ranging from zero to 20 MeV.

In three cases it has been possible from the kinematics of the decay to determine uniquely the charge of the hyperfragment. In this way, two hyperhelium (Bx 3072 and Gö 40) and one hyperlithium (Bx 5021) were identified:

Bx 3072: $\quad {}^5\text{He}_\Lambda \rightarrow \text{p} + \text{d} + 2\text{n}$ or $\text{d} + \text{p} + 2\text{n}$

Gö 40: $\quad \text{He}_\Lambda^4 \rightarrow {}^{1\text{-}2\text{-}3}\text{H} + {}^{1\text{-}2\text{-}3}\text{H} + 2\text{n}$

Bx 5021: $\quad \text{Li}_\Lambda^4 \rightarrow {}^{1\text{-}2\text{-}3}\text{H} + {}^{1\text{-}2\text{-}3}\text{H} + {}^{1\text{-}2\text{-}3}\text{H} + 2\text{n}$.

The other 17 events remain largely undetermined. In 7 cases however limits could be put to the hyperfragment charge (see column 2 in Table I).

4. – Charge distribution.

As a result of the kinetic analysis of some 47 decay stars it has been possible to determine uniquely the charge of 18 hyperfragments. The charge
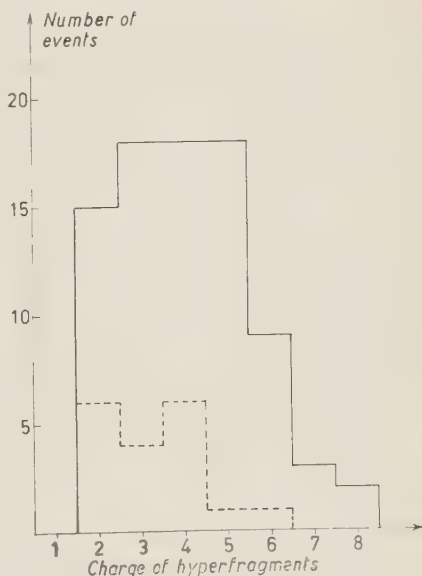

Fig. 1. – Charge distribution of the hyperfragments. - - - hyperfragments of uniquely determined charge. ——— all hyperfragments.

distribution of these events is illustrated in Fig. 1 (dotted line) and exhibited numerically in row (*a*) of Table V.

TABLE V. – *Charge distribution*.

		CHARGE						
		2	3	4	5	6	7	8
Charge uniquely determined (*a*) (18 events)		6	4	6	1	1	—	—
Charge non uniquely determined (19 events)	First approximation (*b*)	2	7	8	2	—	—	—
	Minimal (*c*)	5	5	4	5	—	—	—
	Maximal (*d*)	—	3	5	5	4	2	—
	Intermediate (*e*)	2	5	4	6	2	—	—
Not analyzed (19 events) (*f*)		2	3	2	6	3	2	1
Total (*a*)+(*e*)+(*f*)+DS (*) (*g*) (83 events)		15 (10)	18 (12)	18 (12)	18 (13)	9 (6)	3 (2)	2 (1)
E. M. Silverstein's results (*h*) (142 events)		29	32	43	29	>9		

(*) DS = double centered stars. The bracketed figures refer to the hyperfragments only, neglecting the double centered stars.

In another 19 cases limits could be put to the charge of the hyperfragment. It is rather difficult to get a true idea of the charge distribution of this class of events. Various approaches to its determination have been attempted by different authors ([1,6,7]). As a first rough approximation one may arbitrarily assume, following E. M. SILVERSTEIN ([1]), that half the hyperfragments have a charge equal to the number of prongs of their decay star and that the remaining events have Z equal to one plus the number of prongs. One obtains the charge distribution given in row (*b*) of Table V. From the results of the machine analysis and assigning respectively the minimum and maximum charge to each hyperfragment one gets the charge distribution given in rows (*c*) and (*d*). Moreover, each event yielding many interpretations leading to different Z's, a charge distribution can be obtained by assuming that all the interpretations

([6]) M. BLAU: *Phys. Rev.*, **102**, 495 (1956).
([7]) W. F. FRY, J. SCHNEPS and M. S. SWAMI: *Phys. Rev.*, **106**, 1082 (1957).

are equally probable and by normalizing them to unity. In this manner one determines the *intermediate* distribution numerically illustrated in row (*e*) of Table V. Hereafter we will consider this distribution as the best approximation to the real charge distribution.

In order to get an idea of the overall charge distribution of all the non-mesic hyperfragments produced in K⁻-meson absorption, the data of Fig. 1 should be corrected for the estimated contribution of (*a*) the 10 events for which the kinetic analysis provides only a lower limit of the hyperfragment charge; (*b*) the 9 events which were not analyzed; (*c*) the 27 double centered stars.

Let us consider first classes (*a*) and (*b*). 14, *i.e.* 75% of these hyperfragments contain a short nuclear recoil in their decay star. Owing to our ignorance about the charge carried by these nuclear splinters we consider it too rough an approximation to postulate that the charge distributions of the two types of event are the same. An approach to the determination of the charge distribution of the last events is this: from the machine analysis of the previous 40 events one estimates that the mean charge of the nuclear recoils is 2 or 3. One assumes this to be valid for the remaining hyperfragments. Moreover one supposes that for one half of these events all the other emitted particles are singly charged and that for the other half a doubly charged product is emitted besides the nuclear recoil. In this manner one obtains the charge distribution given in row (*f*) of Table V.

As to the double centered stars, it is clearly hopeless to elaborate any attempt to classification. We will assume their charge distribution to be the same as for the other hyperfragments. So we get the total distribution given in row (*g*) of Table V and illustrated in Fig. 1 (full line).

It must be remarked that the total charge distribution found in the way discussed here should be corrected for 2 experimental biases, *i.e.*:

a) the loss of all hyperhydrogen (neutral mesic and stimulated decays and of all hyperfragments decaying with emission of a single charged particle;

b) the loss of cryptofragments.

At the present, we consider that these biases remain unestimated. Neglecting them, the mean charge of the non-mesic hyperfragments produced in K⁻-meson absorptions is equal to (4.0 ± 0.3). No event of charge greater than 8 has been detected. As a comparison we give in row (*g*) of Table V, the charge distribution of the non-mesic hyperfragments found by E. M. SIL-VERSTEIN [1] studying 4.5 GeV π⁻-meson stars. It must be remarked that the distribution is obtained from a machine analysis similar to ours but disregarding possible emission of two neutrons.

5. – Non-mesic *vs*. mesic decays.

The charge distribution of the mesic hyperfragments reported in paper I is: $Z = 1$, 6 events; $Z = 2$, 13 events; $Z = 3$, 5 events; $Z > 3$, 2 events (*).

Combining this distribution with that for non-mesic decays established above, (row (*f*) of Table V), one obtains as a function of Z the non-mesic to mesic ratio R. The results are listed in row (*a*) of Table VI. In row (*b*) we

TABLE VI. – *Non-mesic to mesic ratio.*

	Charge		
	2	3	> 3
Our results (*a*)	1.2	3.6	25
E. M. Silverstein's results (*b*) (machine analysis)	1.8	8.0	27

give the corresponding values found by E. M. SILVERSTEIN ([4]). There appears to be reasonably good agreement between the two results and the conclusions drawn from this information by E. M. SILVERSTEIN ([4]) about the spin and decay of the Λ^0-hyperon are thus confirmed.

6. – Conclusions.

The production rate of non-mesic hyperfragments including double centered stars in K^--meson absorption at rest (K_σ^--stars only) is found to be $78/2236$, *i.e.*: $(3.5 \pm 0.4)\%$. This value is corrected for the contamination due to Σ^--hyperon absorptions (about 10% of the non definite hyperfragments) but no correction was included to counterbalance the loss of cryptofragments and hyperfragments decaying into a single charged particle. Combining this rate with the one reported in paper I for the mesic hyperfragments, *i.e.*: $(1.2 \pm 0.2)\%$ one gets a total yield of hyperfragments equal to $(4.7 \pm 0.5)\%$. The overall non-mesic to mesic decay ratio R is thus (3.0 ± 0.9). Taking account of the

(*) In paper I, we only reported the charge distribution of the hyperfragments for which the charge has been uniquely determined. To get the distribution given here, we add to this distribution all the other events in the following manner: *a*) one of the 2 events Gö 52 and Bx 6000 is taken as singly charged and the other of charge 2; *b*) event Bx 5018 is assumed to be $^7He_\Lambda$; *c*) event Bx 92 is considered as heavier as lithium; *d*) one of the events Bx 5036 and Bx 10 is taken of charge 3 and the other is assumed to be of charge greater than 3; *e*) the 4 events that have not been analyzed are distributed proportionately to the population of each class.

scanning efficiency for the non-mesic decays, this estimate is a lower limit of R.

The charge distribution of the hyperfragments estimated either from the results of the kinetic analysis of the events is illustrated in Fig. 1. Combining this distribution with the Z-spectrum of the mesic hyperfragments obtained in paper I, one can obtain the non-mesic to mesic decay ratio R as a function of Z. Our results are found to be in good agreement with those of E. M. SILVERSTEIN ([4]).

* * *

Note added in proof.

While this paper was already submitted to publication, we heard of a similar analysis by V. GORGÉ, W. KOCH, W. LINDT, M. NIKOLIC, S. SUBOTIC-NIKOLIC and H. WINZELER (circulated preprint, 28 Juli 1960). Their conclusions are closed to our.

RIASSUNTO (*)

Si espongono i risultati dell'analisi di 78 iperframmenti non mesici, comprese le stelle a due centri, prodotti nell'assorbimento a riposo di mesoni K⁻. Si studia lo spettro degli iperframmenti. 12 decadimenti sono stati identificati univocamente mentre in altri 6 è stata determinata senza ambiguità la carica dell'iperframmento. Per i rimanenti si son potuti porre solo dei limiti alla carica dell'iperframmento. Combinando la distribuzione delle cariche qui determinata con quella ottenuta per gli iperframmenti mesici in una indagine precedente, si può ottenere il rapporto fra i decadimenti non mesici ed i decadimenti mesici in funzione di Z.

Dynamical Consequences of the K^+-K^0 Mass Difference (*).

G. FRYE

Department of Physics, University of California - Berkeley, Calif.

Summary. — The possibility of using the observed K^+-K^0 mass difference for obtaining some information on the $\pi + K$ interaction is investigated. The fraction of the total isotopic vector charge of the K-meson that resides in the two-pion cloud is the crucial parameter. It must have a value greater than unity in order to obtain a simple dynamical origin of the mass difference. Consistency requirements arising from the relation to π-K scattering imply that the fraction must have a value within the interval $-2 \lesssim f_K \lesssim 2$.

We propose to use the K^+-K^0 mass difference [1] for obtaining some information on the dynamical behavior of K-mesons with pions. MATTHEWS and URETSKY [2], and KUANG-CHAO and OGIEVELSKI [3] have suggested a physical mechanism which may cause the neutral member of the spinless isotopic doublet to have a larger electromagnetic self-energy than the charged. First, the neutral particle must be distinct from its antiparticle in order to have a non-vanishing spatial charge distribution [4]. Then it must have more charge concentrated in its short range structure than the charged counterpart. The long-range structure contributes only mildly to the mass difference. We observe, however, that there must be enough charge distributed at a large

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(1) A. H. ROSENFELD, F. T. SOLMITZ and R. D. TRIPP: *Phys. Rev. Lett*,. **2**, 110 (1959); F. S. CRAWFORD jr., M. CRESTI, M. L. GOOD, M. L. STEVENSON and H. K. TICHO: *Phys. Rev. Lett.*, **2**, 112 (1959).
(2) P. T. MATTHEWS and J. L. URETSKY: *Phys. Rev. Lett.*, **3**, 297 (1959).
(3) CHOU-KUANG-CHAO and V. OGIEVELSKI: unpublished.
(4) G. FEINBERG: *Phys. Rev.*, **109**, 1381 (1958).

radius in order to balance the net amount in the « core ». The fringe of the charge distribution is dominated, according to the uncertainty principle, by low-mass intermediate states. The amount of charge that can be found at the large radius of one-half pion Compton wavelength depends on the strength of the matrix element for a K-meson to emit two pions (the emission of one pion is forbidden as a consequence of parity conservation). Some understanding of this matrix element is very important because of the essential role it plays in the long-range K-nucleon interaction. On the basis of the proposed [5] resonance in the $I = 1$, $J = 1$, state of pion-pion scattering we expect it to be quite large. We feel that the K^+-K^0 mass difference provides an interesting context in which to begin the discussion of this matrix element.

The problem of reliably calculating electromagnetic mass differences remains unsolved. The reason is, classically, that the most important contributions to the electrostatic self-energy come from the short-range structure which is dominated by massive, many-particle intermediate states. We shall assume that this complication is represented by an analytic function, which to lowest order in the electromagnetic interaction, is depicted by the Feynman diagram in Fig. 1. [6,7]. The various contributions to this amplitude may be seen by inserting a complete set of physical intermediate states as indicated. We shall be most interested in the one K-meson intermediate state whose effect we can calculate. It contributes a pole with residue equal to the square of the electric form factor [8]. The next intermediate state, K-meson plus one pion, can eventually be calculated by present techniques, but this requires knowledge of several additional

Fig. 1. – The Feynman diagram for Compton scattering of a photon of momentum q on a K-meson of momentum p.

matrix elements. We feel, however, that the dominant part of the higher intermediate states is already included by our method of assuring gauge invariance for the whole amplitude. This was done in an *ad hoc* way on the basis of perturbation theory by including also the local bilinear interaction of the electromagnetic field with the meson field.

(5) W. FRAZER and J. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959).

(6) RIAZUDDIN: *Phys. Rev.*, **114**, 1184 (1959).

(7) M. CINI, E. FERRARI and R. GATTO: *Phys. Rev. Lett.*, **2**, 7 (1959).

(8) G. F. CHEW, R. KARPLUS, S. GASIOROWICZ and F. ZACHARIASEN: *Phys. Rev.*, **110**, 265 (1958); P. FEDERBUSH, M. L. GOLDBERGER and S. B. TREIMAN: *Phys. Rev.*, **112**, 642 (1958).

The electromagnetic mass is obtained from the function represented by Fig. 1. The calculation is done by contracting the photon lines and integrating over all four-momenta q. For the integration we require the square of the electric form factor $F^2(q^2)$ in its unphysical $(q^2 < 0)$ as well as its physical $(q^2 > 0)$ region. The behavior in the unphysical region is obtained by analytic continuation from the physical region where $F(q^2)$ is real. On this basis, we can reduce the four-dimensional momentum space integral to a one-dimensional integral over the invariant momentum square. Before writing down this result, let us decompose the form factor into isotopic scalar and vector parts F^s and F^V. The respective form factors for K^+ and K^0 become

$$F^+ = F^s + F^V,$$

$$F^0 = F^s - F^V.$$

The mass difference between the two particles that arises from the one K-meson intermediate state and the requirement of gauge invariance is given by

(1) $$\delta m_1^2 = \frac{\alpha}{2\pi} \int_0^\infty dq^2\, 4 F^s(q^2)\, F^V(q^2) \left[\frac{q^2}{4m^2} \left(1 + \frac{4m^2}{q^2} \right)^{\frac{3}{2}} - \frac{q^2}{4m^2} \right] \quad (^1).$$

There is no assurance that this integral converges (^{9}). In order to get a finite electromagnetic mass difference we are forced to assume that the form factors fall off rapidly enough for large q^2. This assumption also enables us to write unsubtracted dispersion relations for the form factors (^{8})

(2) $$F^{s,V}(q^2) = \frac{1}{\pi} \int_{(3\mu,2\mu)^2}^\infty \frac{dt\, g^{s,V}(t)}{t + q^2},$$

where $g^{s,V}(t) = \operatorname{Im} F^{s,V}(-t)$. The mass of the pion is μ. The normalization at zero momentum transfer is

$$F^{s,V}(0) \quad \tfrac{1}{2} c.$$

This is important because it requires F^s and F^V to be positive for sufficiently small q^2 (continuity). On the other hand, if δm_1^2 of eq. (1) is to have the same sign as the observed mass difference, $F^s \cdot F^V$ must be negative over a substantial region at larger q^2. These conditions require at least one of the spectral functions to $g^{V,s}$ to be very large in the low-energy range $(t \approx 10\mu^2)$ and

(^{9}) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: Nuovo Cimento, 2, 425 (1955).

have a large contribution of the opposite sign in the high-energy range ($t \approx 200 \, \mu^2$). This situation is qualitatively the same as in the attempted calculations of the proton-neutron mass difference ([7,10]). Thus we conclude that the observed sign of the mass difference is not unacceptable theoretically.

We shall now discuss the spectral function of the isotopic vector form factor on the basis of relativistic dispersion relations ([11]). The only contribution to the spectral function $g^V(t)$ in the interval $4\mu^2 < t < 16\,\mu^2$ comes from the two-pion intermediate state that connects a photon to a $K\bar{K}$ pair

$$(4) \qquad [g^V(t)]_{\pi\pi} = \langle \gamma \,|\, \pi\pi \rangle \langle \pi\pi \,|\, K\bar{K} \rangle \quad .$$

The same two-pion state occurs in the isotopic vector electromagnetic structure of the nucleon ([7]). FRAZER and FULCO ([12]) have shown that a resonance, of suitable position and width, in the $I - 1$, $J - 1$ state of pion-pion scattering gives the pion form factor $\gamma \,|\, \pi\pi$ a behaviour that can provide a natural explanation of large nucleon anomalous magnetic moment and its radius. The interpretation of the vector charge radius remains ambiguous. The reason for this is the large, negative contribution to the spectral function at high energies that is indicated in the calculation of the proton-neutron mass difference. For the strange-meson problem, we assume that the pion form factor $\langle \gamma \,|\, \pi\pi$ is known and we have attempted to infer something about the matrix element $\langle \pi\pi \,|\, K\bar{K} \rangle$. This matrix element is related, by analytic continuation, to the matrix element $\pi K \,|\, \pi K$ of pion – K-meson scattering. The absence of the Yukawa interaction ($\bar{K}K\pi$) makes it possible to do the calculation to within about 20% using S- and P-wave scattering parameters. A large S-wave scattering length (in the $I = \frac{1}{2}$ channel) of about 2 fermis and a strong P-wave resonance at low energy (in the $I = \frac{1}{2}$ channel) gives the spectral function g^V a large magnitude of roughly $3e$ at the peak of the pion-pion resonance ([13]). This contribution alone leads to about twice the value required by eq. (3). An additional large contribution to the spectral function, of the opposite sign and at high energy, can be invoked to correctly normalize the form factor.

([10]) Y. KATAYAMA and M. TAKETANI: to be published.

([11]) G. F. CHEW: *Annual Review of Nuclear Science* (Palo Alto, California, 1959), vol. **9**, p. 29.

([12]) W. FRAZER and J. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959); *Phys. Rev.* (to be published).

([13]) The pion-pion resonance determines the gross shape of the spectral function and provides a factor of ≈ 16 at the peak. Its sign can be reversed by reversing the isotopic spin assignments of the strong S- and P-wave scattering. It is also consistent with crossing symmetry and unitarity to have the same S- and P-wave scattering in both $I = \frac{1}{2}$ and $I = \frac{3}{2}$ states. In this case the two pion contribution to the vector spectral function would vanish.

Reversing the argument, we conclude that the desired high-energy behavior of the spectral function is possible if it has the above large value at low energies.

The gaps left in understanding electromagnetic mass differences prevent us from drawing a firm conclusion about the $\langle \pi\pi | K\bar{K} \rangle$ matrix element. If eq. (1) should indeed describe the mass difference, our argument in favor of a large spectral function and strong pion–K-meson scattering would be stronger and the proposed pion-pion resonance would then play a striking role in accounting for the unusual mass difference. As it is, we shall have to obtain the first quantitative information on our matrix element from the analysis of K-nucleon scattering [14] or from the residue of a pole in the amplitude for the reaction $K^- + p \to \pi^+ + \pi^- + \Lambda^0$ that has been proposed to determine the pΛK parity [15] Finally, it would be of interest to know if a solution of the type discussed here can be found for the self-consistent equation for pion–K-meson scattering of the Chew-Mandelstam type [16].

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I would like to thank Professor ROBERT KARPLUS for many illuminating discussions.

[14] F. FERRARI, M. NAUENBERG and M. PUSTERLA: *University of California Radiation Laboratory Report* UCRL-8985 (unpublished).

[15] G. F. CHEW and F. LOW: *Phys. Rev.*, **113**, 1640 (1959); L. B. OKUN' and I. YA. POMERANCHUK: *Žur. Éksp. Theor. Phys. U.S.S.R.*, **36**, 300 (1959) (translation: *Soviet Phys. - JETP*, **9**, 207 (1959)).

[16] G. F. CHEW and S. MANDELSTAM: *University of California Radiation Laboratory Report*, UCRL-8728 (unpublished).

RIASSUNTO

Si esamina la possibilità di ottenere informazioni sull'interazione π-K dall'esame della differenza di massa fra K^+ e K^0 osservata sperimentalmente. La frazione della parte vettoriale della carica del mesone K che risiede nella nuvola dei due pioni costituisce il parametro determinante del problema. Tale parametro deve essere maggiore di uno affinchè si possa dare una semplice interpretazione dinamica alla differenza di massa. Relazioni di congruenza che derivano dallo studio dello scattering π-K impongono che la frazione di carica debba avere un valore interno all'intervallo $-2 \leqslant f_K \leqslant 2$.

$|\Delta S| = 1$, $|\Delta I| = \frac{1}{2}$ Rules with Two Charged Intermediate Boson Fields.

B. d'Espagnat

CERN - Geneva (*)

Summary. — The rules $|\Delta S| = 1$ for non leptonic decays and $|\Delta I| = \frac{1}{2}$ are accounted for by a theory in which weak interactions are due to charged intermediate bosons that have no neutral counterpart. Leptonic decays of strange particles where the total charge of the leptons is zero are thereby forbidden, in accordance wiht experiment. Observable differences with the prediction of the Lee and Yang theory are discussed. The theory gives arguments in favour of the existence of a decay mode of the cascade particle into a nucleon and leptons with a coupling constant equal to that of the leptonic decay mode of the Λ, *i.c.* with a decay rate about twenty times smaller than the rate for the normal Ξ-decay.

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1. – Introduction.

In a recent paper (¹), Lee and Yang investigated the consequences of the following three propositions:

    i) All weak interactions are transmitted through an intermediate boson field w.

    ii) The K_1^0, K_2^0 mass difference is small ($< 10^{-5}$ eV), or, in other words there are no $\Delta S = \pm 2$ among the usual non leptonic weak interactions.

    iii) The $|\Delta I| = \frac{1}{2}$ rule holds for the strangeness non conserving decays of particles, where I is the total isospin of the strangely interacting particles

This investigation led them quite naturally to a scheme in which

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(¹) T. D. Lee and C. N. Yang: to be published.

a) There exist two complex intermediate boson fields w and w^0, representing altogether four kinds of particles w^+, w^-, w^0_a, w^0_b which bear some similarity with the $K^+ K^- K^0_1 K^0_2$ complex.

b) These fields have dual isospin properties in the sense that they behave as components of an isospinor when coupled to the strangeness-non-conserving current and as an isoscalar and the components of an isovector when coupled to the strangeness-conserving current ([2]). For that reason they are called schizons.

Although the schizon scheme is attractive it is not a unique consequence of propositions i), ii), iii). In the first place it can be modified and generalized by keeping assumption a) and dropping assumption b) so that the w's are not schizons any more. This rather trivial generalization is briefly described in Section **4** together with its experimental implications. The main purpose however, of the present paper is to describe an altogether different theory which assumes the existence of four charged intermediate bosons w^+, w^-, v^+, v^- (*i.e.* two complex boson fields) and of *no* neutral ones. It is shown below (Section **2**) that propositions i), ii) and iii) are compatible with such an assumption, iii) being either an exact statement or a statement which is exact only in the limit of an infinite intermediate boson mass, depending on the choice of the parameters. In the latter case the $|\Delta I| \geqslant \frac{3}{2}$ admixtures due to a finite intermediate boson mass are discussed and shown to be small for reasonable values of the parameters involved.

The main interest of a theory based on charged intermediate bosons only is that it implies the validity of a fourth proposition:

iv) All decays that involve one lepton pair among the decay products are strictly forbidden apart from electromagnetic effects when the total charge of this pair is zero.

This fact is important, for proposition iv) is indeed as well rooted in experimental facts as proposition iii) for instance. In order to keep this property in mind, we suggest that the w and v particles should be called « vetons ».

LEE and YANG ([1]) have described a number of observable predictions of the schizon theory. Many of these are the same in the veton theory but some others are different and could serve as experimental tests between the two schemes. A detailed comparison of the predictions of the two theories is made in Section **4**.

([2]) In this respect also, the Lee and Yang w particles are rather similar to the K's, particularly if one thinks of the « M space » interpretation of the $|\Delta I| = \frac{1}{2}$ rule (B. D'ESPAGNAT, J. PRENTKI and A. SALAM: *Nucl. Phys.*, **5**, 447 (1958)).

2. – Non-leptonic interactions.

The simplest way to write down a four-baryon interaction that satisfies the $|\Delta I| = \frac{1}{2}$ rule is to multiply an isoscalar current with a current which is the component of an isospinor; here it is, moreover, required that these currents should be charged. This leads in a straight-forward way to the weak interaction Lagrangian

$$(1) \qquad L'' = (\bar{\Xi} \cdot N)(\bar{p}\Lambda) + \text{h. c.},$$

where (as everywhere below) the factors $i\gamma_\mu(1+\gamma_5)$ corresponding to the $V-A$ coupling scheme are assumed to be present inside each bracket and where $(\bar{\Xi} \cdot N)$ means the scalar product of the two isospinors $(\bar{\Xi}_0, \bar{\Xi}_-)$ and $\begin{pmatrix} p \\ n \end{pmatrix}$:

$$(2) \qquad (\bar{\Xi} \cdot N) = (\bar{\Xi}_0 p) + (\bar{\Xi}_- n),$$

$(\bar{\Xi} \cdot N)$ is an isoscalar and $(p\Lambda)$ is a component at an isospinor so that (1) gives a strict $|\Delta I| = \frac{1}{2}$ rule. This is true to any order in the strong interaction coupling constants and implies no assumption whatsoever as to the relative values of these coupling constants or of the N and Ξ masses. Eq. (1) on the other hand involves no neutral current. These simple facts make (1) suitable as a starting point for the development of the theory and indeed are the basis of the two schemes A and B described below. Eq. (1) however involves no strangeness-conserving current so that some transformations or adjunctions are necessary before it can be used.

2˙1. *First simple model.* – Eqs. (1) and (2) give

$$(3) \qquad L'' = (\bar{\Xi}_0 p(\bar{p}\Lambda) + (\bar{\Xi}_- n)(\bar{p}\Lambda) + \text{h. c.}.$$

By taking advantage of the identity

$$(4) \qquad (\bar{\psi}_1 \psi_2)(\bar{\psi}_3 \psi_4) = (\bar{\psi}_3 \psi_2)(\bar{\psi}_1 \psi_4)$$

one can now exchange $\bar{\Xi}_-$ and $\bar{p}$ in the second term of (3). This gives

$$(5) \qquad L'' = (\bar{\Xi}_0 p)(\bar{p}\Lambda) + (\bar{p}n)(\bar{\Xi}_-\Lambda) + \text{h. c.}.$$

L'' is a part of

$$(6) \qquad L' = (a\bar{p}n + a^{-1}\bar{\Lambda}\bar{\Xi}_-)(a\bar{n}p + a^{-1}\bar{\Xi}_-\Lambda) + (b\bar{p}\Xi_0 + b^{-1}\bar{p}\Lambda)(b\bar{\Xi}_0 p + b^{-1}\bar{\Lambda}p),$$

a and b being arbitrary real coefficients, and indeed L'' is the only part of L' which can contribute to the non leptonic decays of strange particles, for the other terms in L' all have $\Delta S = 0$. Therefore L' also satisfies a strict $|\Delta I| = \frac{1}{2}$ rule. To the first order in L' there is, moreover, no $\Delta S = \pm 2$ transition in agreement with recent experimental results ([3]). Thus L' is consistent with both propositions ii) and iii).

The assumption that weak interactions are due to intermediate bosons (proposition i)) can now be easily introduced. Let us define two complex vector fields w_μ and v_μ which describe two charged bosons of each charge (the index μ is dropped in what follows) and let us replace L' by

$$(7) \qquad L = [f_1(\bar{p}n) + f_2(\overline{\Lambda}\,\Xi_-)]_w + [f_1'(\bar{p}\,\Xi_0) + f_2'(\bar{p}\Lambda)]v + \text{h. c.},$$

with the condition

$$(8) \qquad\qquad f_1 f_2 = f_1' f_2', \qquad m_w = m_v,$$

which ensures that, in the limit $m_w = m_v = \infty$, eq. (7) produces the same effects as (6). In the limit $m_w = m_v = \infty$, (7) therefore describes non-leptonic
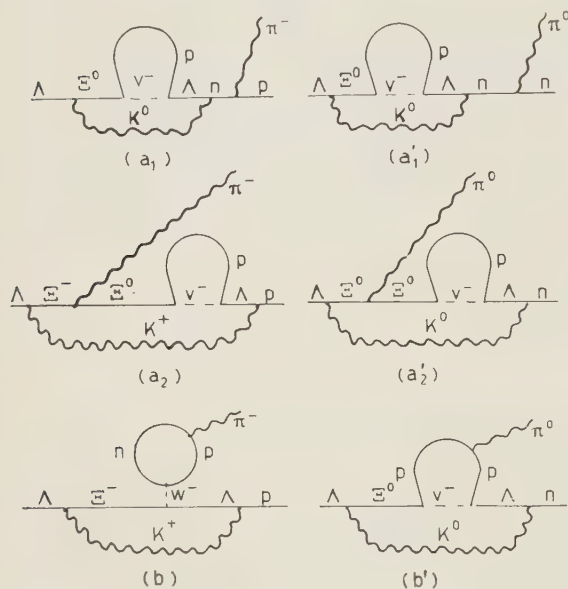

Fig. 1. – Some lowest order graphs in Λ-decay, the weak interactions being described by (7).

decays that obey strict $|\Delta I| = \frac{1}{2}$, $|\Delta S| = 1$ rules. If, on the other hand, the masses m_w and m_v are chosen large but finite, the $|\Delta S| = 1$ rule remains strictly valid as before but, in contrast to what happens in Lee and Yang's case, the $|\Delta I| = \frac{1}{2}$ rule becomes approximate (even with absence of electromagnetic interactions). This is due to the fact that the finite intermediate boson masses make L' a non-local interaction, with the effect that there is no more a strict identity between (3) and (5).

As an illustration of how the weak interaction effec-

([3]) F. MULLER, R. W. BIRGE, W. B. FOWLER, R. H. GOOD, W. HIRSCH, R. P. MATSEN, L. OSWALD, W. M. POWELL, H. S. WHITE and O. PICCIONI: *Phys. Rev. Lett.*, **4**, 418 (1960).

tively operates in a particular case, Fig. 1 shows some of the lowest order graphs pertaining to Λ-decay. It is interesting to notice that graphs of types (a) in Fig. 1 give a strict $|\Delta I| = \frac{1}{2}$ rule irrespective of the veton mass while graphs of type (b) give a $\Delta I = \frac{3}{2}$ admixture which vanishes only for infinite veton masses. A similar distinction holds of course for higher order graphs. Under such circumstances a reliable estimate of the $\Delta I = \frac{3}{2}$ admixture as a function of the intermediate boson mass is not easily obtained. It seems, however, that for a mass of the order of the nucleon mass this admixture is presumably rather small ([4]), even in the simplified version here described. One can, however, make it even smaller by generalizing the theory in the manner described below.

2'2. *Second simple model.* – Another, remarkably simple model ([5]) which satisfies propositions i), ii), iii) and iv) is obtained directly from (1), by assuming that interaction (1) is due to an intermediate boson v. It is

$$(9) \qquad f_1(\overline{p}n)w + \{f_3'[(\overline{p}\,\Xi_0) + (\overline{n}\,\Xi_-)] + f_2'(\overline{p}\Lambda)\}v + \text{h. c.} .$$

Indeed, proposition iii) is in this model an *exact* statement for any value of m_v: this follows from the fact that, if v is considered as an isoscalar ([6]), the terms which multiply f_3' and f_2' are an isoscalar and an isospinor component respectively. As for the term in f_1 its necessity appears in connection with leptonic couplings and the description of β and pion decays.

2'3. *A more general version of the theory.* – A more general version of the veton theory, which admits (7) and (9) as special cases, is obtained by applying

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([4]) Preliminary lowest order calculations gave roughly 50% $\Delta I = \frac{3}{2}$ admixture under the most unfavourable and highly unlikely assumptions that all graphs (a) of Fig. 1 are negligible, that $m_v = 0$ and that the cut-off is infinite.

([5]) If the existence of an unstable and (e.g. positively) charged baryon X^+ having hypercharge 2 and isospin zero is postulated, then propositions i) to iv) can be made exact statements by choosing a weak interaction Lagrangian of the type

$$f(\overline{p}n)w + [f'(\overline{X}_+\Lambda) + f''(\overline{p}\Lambda)]v + \text{h. c.} .$$

This interesting fact, which was pointed out by Dr. PRENTKI in a private discussion of the ideas here described, has led the author to consider the simple interaction Lagrangian described by (9) and by scheme B below, which has the same property without having the disadvantage of postulating the existence of a new particle. Our thanks are due to Dr. PRENTKI for this suggestion and other critical remarks on the content of the theory.

([6]) An isopseudoscalar if reflections are considered (B. D'ESPAGNAT and J. PRENTKI: *Nucl. Phys.*, **1**, 33 (1956)).

(4) to a fraction only of the second term in (3) and by applying to the resulting expression the same arguments as those which lead from (5) to (7). The result is

(10) $L = [f_1(\overline{p}n) + f_2(\overline{\Lambda}\,\Xi_-)]w + [(f_1' + f_3')(\overline{p}\,\Xi_0) + f_3'(\overline{n}\,\Xi_-) + f_2'(\overline{p}\Lambda)]v + \text{h.c.}$

with

(8) $f_1 f_2 = f_1' f_2'$

(10) leads to an exact $|\Delta I| = \frac{1}{2}$ rule for an infinite veton mass. It reduces to (7) for $f_3' = 0$ and to (9) for $f_2 = f_1' = 0$. Eq. (9) gives no $|\Delta I| \geqslant \frac{3}{2}$ admixture whatsoever. Therefore, if y is, for a given process, the ratio (discussed in Section 2·1) of $|\Delta I| = \frac{3}{2}$ to $|\Delta I| = \frac{1}{2}$ matrix elements that follow from (7) alone, this same ratio takes, with Lagrangian (10), the (smaller) value

(11) $$y' = \frac{f_1'}{f_1' + f_3'\alpha}\, y \,,$$

where α is the ratio of the $|\Delta I| = \frac{1}{2}$ matrix elements produced by (7) and (9) respectively when $f_3' = f_1'$. As a matter of fact $\alpha \approx 1$ whenever y is not too large. Eq. (11) is used below for a rough quantitative estimate of the $|\Delta I| \geqslant \frac{3}{2}$ admixture with special choices of the parameters.

3. – Leptonic couplings and specification of coupling constants.

The leptonic interactions of vetons are given by the Lagrangian

(12) $L_l = [f_e(\overline{\nu}e_-) + f_\mu(\overline{\nu}'\mu_-)]w + [f_e'(\overline{\nu}e_-) + f_\mu'(\overline{\nu}'\mu_-)]v + \text{h. c. (}^7\text{)} \,.$

The leptonic decays of strange particles then take place through the emission of a virtual veton by (10) followed by its absorption by (12). This has a first important consequence: owing to the fact that both $(\overline{\Lambda}\,\Xi_-)$ and $(\overline{p}\Lambda)$ are components of isospinors, all decays of strange particles which involve a lepton pair among the decay products obey a strict $|\Delta I| = \frac{1}{2}$ rule, where I is the total isospin of the strongly interacting particles in the initial and final states. They therefore also obey the $\Delta Q/\Delta S = 1$ rule which is a consequence of the former (and of the fact that the lepton pair is charged). The situation is thus identical in this respect to the one which prevails in the schizon theory. The numerous consequences of these rules have been explored before, espe-

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(7) For the sake of completeness let us mention here that v and v' could be exchanged in *one* of the square brackets of (12).

cially by Feynman and Gell-Mann ([8]) and by Okubo, Marshak, Sudarshan, Teutsch and Weinberg ([9]).

The next problem is to choose the coupling constants in (10) and (12) in such a way as to correctly reproduce the experimental data. The most important of these is the near equality between β-decay, π-decay and μ-decay coupling constants which gives

$$(13) \qquad f_1 f_e = f_1 f_\mu = f_e f_\mu + f_e' f_\mu' .$$

Another important fact is the existence of leptonic decays of K particles and the existence and relative scarcity of

$$(14) \qquad \Lambda \to p + e^- + \bar{\nu} .$$

These facts are compatible with relations

$$(15) \qquad f_2 f_e + f_2' f_e' \approx f_2 f_\mu + f_2' f_\mu' \approx 0.25 f_1 f_e .$$

Finally, the $\Delta S = 2$, leptonic decays

$$(16) \qquad \begin{cases} \Xi^0 \to p + e^- + \bar{\nu} \\ \Xi^- \to n + e^- + \bar{\nu} \end{cases}$$

have not been observed, although phase-space calculations show that their rate should be almost equal to the observed Ξ-decay rate if the relevant coupling constant were the same as in β-decay's. Considering that nearly 20 Ξ-particles have been observed up to now, this gives the two inequalities

$$(17) \qquad (f_1' + f_3') f_{e,\mu}' \leqslant 0.25 f_1 f_e ,$$

where $f_{e,\mu}'$ stands for f_e' and f_μ'.

It is rather unfortunate that the experimental information on processes (16) is so meagre. In what follows two models are described which correspond to the two extreme assumptions (A) that (16) is completely forbidden, and (B) that (17) is very nearly an equality.

Scheme A. – Here we assume that neither (16) nor the corresponding μ-decay can happen, *i.e.* that

$$(18) \qquad f_e' = f_\mu' = 0 .$$

([8]) R. P. Feynmann and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).

([9]) S. Okubo, R. E. Marshak, E. C. G. Sudarshan, W. B. Teutsch and S. Weinberg: *Phys. Rev.*, **112**, 665 (1958).

(13) then gives

(18') $f_1 = f_e = f_\mu$

and with (8) and (15), (10) becomes

(19) $L = f_1[(\overline{p}n) + 0.25\,(\overline{\Lambda}\,\Xi\,v)]w + [(f_1' + f_3')(\overline{p}\,\Xi_0) + f_3'(\overline{n}\,\Xi_-) + 0.25\,f_1^2 f_1'^{-1}(\overline{p}\Lambda)]_- + \text{h.c.}$

The conventional β and μ-decay coupling constants g_F are then essentially proportional to f_1^2 while the Fermi coupling constant g_F' relative to non leptonic decays is proportional to

$$0.25\,(1 + f_3' f_1'^{-1})f_1^2\,.$$

In the present state of knowledge this gives only a very rough upper limit for $f_3' f_1'^{-1}$, which one may tentatively choose as

(20) $0 < f_3' f_1'^{-1} < 10\,.$

Eq. (11) then shows that, with the estimate $y \ll 50\%$ obtained in Section 2·1 ([4]), the upper limit of the relative amount, y', of $|\Delta I| \geqslant \frac{3}{2}$ admixture produced by Lagrangian (19) when the veton mass is finite can be very small: it is less than 10% for $f_3' f_1'^{-1} \sim 4$, i.e. for $g_F' \sim g_F$.

Similar considerations apply to possible extensions of scheme A that would make it consistent with the conserved current hypothesis ([8]). If this hypothesis is correct, then supplementary terms are needed in (10) which, in contrast with the previous case, give $|\Delta I| \geqslant \frac{3}{2}$ admixtures that do not vanish even for an infinite veton mass. The relative amount y' of such admixtures still satisfies, however, relation (11) (where y is the value of y' for $f_3' = 0$) as is apparent from the fact that (9) leads to a strict $|\Delta I| = \frac{1}{2}$ rule whatever modifications are made to the strangeness-conserving current coupled to w. Under these conditions y' can be made reasonably small provided y is not too large. This, in turn, is presumably the case, also here, whenever graphs of type « a » in Fig. 1 are not negligible as compared to graphs of type « b ». It may finally be noticed that y is particularly small if π, Σ and Κ particles are considered as compounds whose couplings to the Ξ are only moderately strong: the conserved current hypothesis is then satisfied by the simple adjunction of a $(\overline{\Xi}_0 \Xi_-)$w interaction whose effects, independently of all other considerations, are intrinsically small. The foregoing discussion shows that scheme A can be made to fit all presently established experimental facts by suitable choices of the parameters and with any reasonable choice for the value of the veton mass. It also shows, though merely on the basis of rather qualitative arguments, that it could be adapted to the conserved current hypothesis. It

should be recognized, however, that this latter point is open to discussion for, if the conserved current hypothesis would prove to be strictly correct, scheme A would look somewhat artificial.

Scheme B. – Here we assume that (17) is nearly an equality (which means that decays (16) should be observed in the near future). The leptonic decays of strange particles can then be entirely attributed to the v-particle acting as intermediate, or, in other words, one may put

(21) $$f_1' = f_2 = 0 .$$

L then takes the simple form given by eq. (9). Immediate consequences of this are that the $|\Delta I| = \frac{1}{2}$ rule is strictly valid (apart from electromagnetic corrections):

   a) for any (finite as well as infinite) value of the veton mass;

   b) also if the conserved current hypothesis is made $((\bar{p}n)$ in (9) being replaced by J, the strangeness conserving current whose vector part is divergenceless).

Relations (15), (17) and (21) give

(22) $$\begin{cases} f_e = f_\mu , \qquad f_e' \approx f_\mu' , \\ f_1 \approx (f_e^2 + f_e'^2)/f_e , \\ f_2' \approx f_3' \approx (f_e^2 + f_e'^2)/4f_e' \end{cases}$$

and the condition that the Fermi coupling constant for non leptonic decay $g_F' \propto f_2'^2$ and the conventional β-decay coupling constant $g_F \propto f_1 f_e$ should be of comparable orders of magnitude sets a further, loose, requirement which can be satisfied in a variety of ways. As an example, we mention here the simple choice: $f_e = f_e' = f_1/2 = 2f_2'$.

Scheme B is attractive in that:

   a) it is strictly consistent with the propositions i), ii), iii) and iv) stated in the introduction, for any veton mass value;

   b) its generalization, so as to incorporate the conserved current hypothesis, is trivial and finally;

   c) it can be easily generalized and given interesting symmetries.

As an example of the latter point, one may mention the following generalization of the non leptonic interaction Lagrangian

(23) $$L = f_1 Jw + f_2'[(\bar{N}\cdot\Xi) + (\bar{N}\cdot Z) + (\bar{Y}\cdot\Xi)]v + \text{h. c.} ,$$

where J is the strangeness-conserving current and Y, Z are the two well-known doublets [10] made of Σ's and Λ.

On the other hand, scheme B definitely predicts the existence of the decay modes (16), the decay rates for (16) and for the leptonic Λ-decay being both weaker than the values predicted from universality by roughly the same factor.

4. – Observable differences between the veton and schizon theories.

4'1. *Pseudo-schizons*. – Before discussing the similarities and differences between the veton and schizon theories, it is perhaps appropriate to mention very briefly two rather trivial generalizations of the latter. One of them is to rewrite formulae (23) and (24) of Lee and Yang's paper as

$$(24) \qquad f_2(\overline{\Lambda p})w^* + f_2'(\overline{\Lambda n})w^{0*} + \text{h. c.} ,$$

$$(25) \qquad f_1(\overline{n}p)w^* - \tfrac{1}{2}f_1'[(\overline{p}p) - (\overline{n}n)]w^{0*} + \text{h. c.} ,$$

with

$$(26) \qquad f_1 f_2 = f_1' f_2' .$$

The other one is to add to these Lagrangians a term

$$(27) \qquad f_3[(\overline{p}\,\overline{p}) + (\overline{n}n)]w^{0*} + \text{h. c.} .$$

Both these generalizations can be made simultaneously. Each of them destroys the isovector behaviour of the « schizons » when coupled to the strangeness-conserving current (with the result that they are not true « schizons » any more) but the validity of propositions i), ii) and iii) is identically preserved, for any value of m_w. The proof of this statement (particularly concerning iii)) is elementary: as regards the first generalization ($f_i \neq f_i'$) it relies on the fact that in any $\Delta S = \pm 1$ decays a charged w which is emitted with f_1 is necessarily reabsorbed with f_2 and conversely so that only the product $f_1 f_2$ comes in (and, similarly, the product $f_1' f_2'$ for w⁰); as regards the second generalization it is simply adding an isospinor component to a Lagrangian which was already constructed to be an isospinor component. The Lee and Yang scheme so enlarged will be called, for brevity, the pseudo-schizon theory.

4'2. *Experimental predictions with pseudo-schizons*. – These are almost exactly the same as with the schizons. Indeed the only real change is that the relations between w$^\pm$ and w$_a^0$, w$_b^0$ decay rates into pions and kaons (eqs. (45) and (47) of ref. [1]) get lost. Also the upper limits on the lepton

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[10] M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).

couplings to w⁰ have of course to be somewhat modified. Apart from this all the predictions derived in ref. (^{1}) from the schizon theory are valid for the pseudo-schizon theory (including those which stem fron the fact that the charged current J is a component of an isovector). This generalization of the Lee and Yang scheme is not therefore of a critical interest for experiments.

4˙3. *Experimental predictions with retons.* – If scheme B (Section **3**), is correct Ξ-decays into nucleons and leptons should be observed in the near future. This is an essential difference with the schizon scheme which completely forbids such modes of decay (at least to the 1-st order in weak interactions which are the only relevant ones in such problems). As such experiments are comparatively easy it is not necessary to discuss any further the more subtle differences between the predictions of the schizon theory and of the veton theory in its version « B ». The remainder of this section is thus concerned with observable differences between the schizon theory and the veton theory in its version « A », *i.e.* when such modes of Ξ-decay are forbidden by both theories.

α) **Processes involving no real veton.** As long as no (real particle) veton is created, the predictions of the veton scheme A are essentially described by propositions i) to iv) of Section **1**, *i.e.* are substantially the same as those of the schizon theory. The two main differences are

a) that the veton theory strictly forbids any strange particle decay with a lepton pair of total charge zero among the decay products;

b) that, for non-leptonic decays, scheme A predicts some amount of non-electromagnetic $|\Delta I| \geqslant \frac{3}{2}$ admixture when the veton mass is finite and/or when the conserved current hypothesis is made (as discussed in Section **3**).

Also in the class of phenomena with no real vetons are the decays

(28)
$$\begin{cases} \Sigma^- \to \Lambda + e^- + \bar{\nu} \\ \Sigma^+ \to \Lambda + e^+ + \nu \end{cases}$$

and the inverse μ capture processes

(9)
$$\begin{cases} \nu' + n \to \mu^- + n + \pi^+ \\ \nu' + p \to \mu^- + p + \pi^0 \\ \nu' + p \to \mu^- + p + \pi^+ \end{cases}$$

and, more generally,

(30)
$$\begin{cases} \nu' + n \to \mu^- + \Gamma, \\ \bar{\nu}' + p \to \mu^+ + \Gamma', \end{cases}$$

where Γ and Γ' are complexes of strongly interacting particles with total strangeness zero. All these phenomena proceed through w only and, more precisely, through the term $f(\overline{p}n)w$ in (10), where $(\overline{p}n)$ is a component of an isovector: therefore all the conclusions drawn in ref. (^{1}) for such processes using the schizon theory are also valid in the veton theory, schemes A and B. The detailed experimental study of such processes would therefore offer no clues for a discrimination between the two theories. The same remark holds for the reactions

$$(31) \quad \begin{cases} \overline{\nu}' + n \rightarrow \mu^+ + \Sigma^- & \text{or} \quad \mu^+ + \Lambda + \pi^- \\ \overline{\nu}' + p \rightarrow \mu^+ + \Sigma^0 & \text{or} \quad \mu^+ + \Lambda + \pi^0, \end{cases}$$

which proceed through the isospinor components $(\overline{\Lambda \Xi_-})$, $(\overline{p}\Lambda)$ and therefore obey $|\Delta I| = \frac{1}{2}$ exactly.

β) **Processes involving the creation of vetons as real particles** (scheme A). If the vetons are created using a beam of ν', or if they are created in a process which involves strongly interacting particles and has $\Delta S = 0$, S being the total strangeness of the initial and final systems of strongly interacting particles, then (18) and (19), respectively, show that only type w vetons are created. If on the other hand these vetons are created in a process which involves strongly interacting particles and has $\Delta S = \pm 1$, both w and v can be created. Examples of such processes are

$$(32) \qquad\qquad \pi^- + p \rightarrow \Sigma^- + (w^+ \text{ or } v^+),$$

$$(33) \qquad\qquad K^+ + p \rightarrow p + (w^+ \text{ or } v^+).$$

Finally, if these vetons are created in a similar process with $\Delta S = \pm 2$ such as

$$(34) \qquad\qquad K^+ + p \rightarrow \Sigma^+ + v^+$$

only v vetons can be created.

On the other hand, the w-decay modes are $e + \nu$, $\mu + \nu$, 2π, 3π, $K + \pi$, $K + \gamma$ etc., whereas the v-decay modes are essentially $K + \pi$, $K + \gamma$, $K + 2\pi$ etc. and possibly $K + K$. The branching ratios for the decay of vetons of a given charge are therefore very different according to whether these vetons are all w all v or a mixture of both, i.e. from what was said above, depending on their mode of production (^{11}). Nothing of that sort takes place in the schizon

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(^{11}) We are indebted to Professor YANG for this remark.

theory where there is only one type of charged schizon: this offers therefore in principle a means of discriminating between the schizon and the veton pictures. In practice let us consider for instance the processes with « apparent » change of strangeness (see ref ([1]) section **10**)

$$\text{K}^+ + \text{p} \rightarrow \Lambda + \text{K}^+ + \text{w}^+ \rightarrow \begin{cases} \Lambda + \text{K}^+ + \text{e}^+ + \nu \,, \\ \Lambda + \text{K}^+ + \mu^+ + \nu' \,, \\ \Lambda + \text{K}^+ + \pi^+ + \pi^0 \,, \\ \Lambda + \text{K}^+ + \pi^+ + \gamma \,. \end{cases}$$

(35a)
(35b)
(35c)
(35d)

In the schizon theory the final states can only be obtained from the initial state through the intermediate channel $\Lambda + \text{K}^+ + \text{w}^-$, as shown in (35) (creation of a real w^+ which subsequently decays). The branching ratios for (35a, b, c, d) are therefore the same as those of

$$\nu' + \text{p} \rightarrow \mu^- + \text{n} + \text{w}^+ \rightarrow \begin{cases} \mu^- + \text{n} + \text{e}^+ + \nu \\ \mu^- + \text{n} + \mu^+ + \nu' \\ \mu^- + \text{n} + \pi^+ + \pi^0 \\ \mu^- + \text{n} + \pi^+ + \gamma \end{cases}$$

(36a)
(36b)
(36c)
(36d)

In the veton theory, on the other hand, the final states in (35c, d) can also be obtained through

$$\text{K}^- + \text{p} \rightarrow \Lambda + \pi^+ + \text{v}^+ \rightarrow \begin{cases} \Lambda + \pi^+ + \text{K}^+ + \pi^0 \\ \Lambda + \pi^+ + \text{K}^+ + \gamma \,. \end{cases}$$

(37c)
(37d)

These contributions add to (35c, d) with the result that the net branching ratios for the overall processes described by $\text{K}^+ + \text{p} \rightarrow$ final states of (35) becomes different from what they are in (36). The experimental discrimination between the two theories consists therefore in this case in picking out the events $\text{K}^+ + \text{p} \rightarrow$ all final states of (35) (which, if they exist, should be discriminated from ordinary interactions through the change of strangeness they exhibit) and in comparing their relative amounts to the relative amounts of $\nu' + \text{p} \rightarrow$ all final states of (36). The same method can be used in pp collisions (replace $\text{K}^+ + \text{p}$ by $\text{p} + \text{p}$ and Λ by 2Λ in (35) and (37)).

Another difference between the schizon and veton theories is that apparent processes such as

$$\begin{cases} \text{K}^+ + \text{p} \rightarrow \Sigma^+ + \text{K}^+ + \pi^0 \\ \text{K}^+ + \text{p} \rightarrow \Sigma^+ + \text{K}^+ + \pi^+ \end{cases}$$

(38)

which also fall in the class of processes with apparent change of strangeness discussed in ref. ([1]), Section **10**, can happen if the veton theory is correct (the intermediate real state is then $\Sigma^+ + v^+$) but are forbidden in the schizon theory The same is true for

$$(39) \qquad p + p \to \begin{cases} \Lambda + \Sigma^+ + K^+ + \pi^0 \\ \Lambda + \Sigma^+ + K^0 + \pi^+ \end{cases}$$

Observation of such « apparent » processes would therefore discriminate in favour of the veton theory.

In the third place the veton theory can, as is easily checked, induce no process with apparent $\Delta S = \pm 2$, whereas the schizon theory can, due to the creation of a real w_n^0 (see eq. (52) ref. ([1])). Observation of such processes would thus discriminate in favour of the schizon theory: so would, of course, any other conceivable experiment showing that a w^0 exists.

Finally, one may mention the fact that the branching ratio for v and w non leptonic decays are the same as those derived in ref. ([1]) for the charged schizon. This follows from the fact that $\overline{p}n$, $\overline{\Lambda}\Xi_-$, and $\overline{p}\Lambda$ are components of isotopic vectors or spinors. If $m_v > 2m_K$, v however possesses a decay mode of its own which is

$$v^+ \to K^+ + K^\cdot .$$

The conclusion of this subsection is that at least five experimental means of discriminating between the schizon and veton theories can be found. These are

a) a search for Ξ decays into a nucleon and leptons;

b) a search for leptonic decays with a leptonic pair of total charge zero;

c) a comparison between the branching ratios of (35) and (36);

d) a search for some special processes involving « apparent » non conservation of strangeness, such as (38) or (39), and

e) a search for « apparent » $|\Delta S| = 2$ processes.

5. – Other forms.

One may ask at this stage whether the two versions A and B above of the veton theory are just two arbitrary examples taken out of a large set of qualitatively different Lagrangians that would all satisfy propositions i) to iv) or whether the theory with charged intermediate bosons only is, apart from various possible choices of the parameters, reasonably unique.

The discussion of this problem is rather involved in the case that the $|\Delta I| = \frac{1}{2}$ rule is required only for infinite veton masses because one has then the supplementary freedom of transforming any given term by means of (4)

as is done for instance in scheme A. In spite of this, however, the only alternative to scheme A that we found ([12]) is obtained by applying the procedure of Section 2·1 to $(\bar{\Xi}N)(\bar{\Lambda}\Xi_-)$ instead of (1). The resulting scheme leads, however, to difficulties as regards the ratio of β-decay to μ-decay coupling constants.

In the case that the $|\Delta I| = \frac{1}{2}$ rule is required to hold exactly for any value of the veton mass as in scheme B, the discussion of possible alternatives can be made more rigorous. The $\Delta S = \pm 1$ currents must be isospinor components, because of iii) for leptonic decays; the $\Delta S = 0$, ± 2 currents which combine with these must then have either $I = 0$ or $I = 1$, because of iii) for non leptonic decays. Being charged, they satisfy moreover to

$$\Delta Q = \Delta I_3 + \frac{1}{2}\Delta U = \pm 1 .$$

If $I = 0$ this relation gives $\Delta U = \pm 2$: the only possible choice is then $(\bar{\Xi} \cdot N)$, with possible adjunction of $K^T \tau_2 K$. If $I = 1$ the component with $I_3 = 0$ has again $\Delta U = \pm 2$ (as has $(\bar{\Xi} \cdot N)$) but then the component with $I_3 = \mp 1$ would have $\Delta Q = 0$. This shows that $(\bar{\Xi} \cdot N)$ is essentially the only possible $\Delta S \neq \pm 1$ current that can be used in $|\Delta I| = \pm \frac{1}{2}$ interactions. As for the $\Delta S = \pm 1$ current, it can be any combination of the isospinor components

$$(\bar{p}\Lambda) ; \quad (\bar{\Lambda}\Xi_-) ; \quad (\bar{p}\Sigma_0) + \sqrt{2}(\bar{n}\Sigma_-) ; \quad -(\bar{\Sigma}_0\Xi_-) + \sqrt{2}(\bar{\Sigma}_+\Xi_0) .$$

This gives a qualitative description of all the possible generalizations of scheme B.

One of the most attractive among such generalizations is probably (23), which shows that also in the veton theory particles and Lagrangians can be given attractive formal symmetries.

The formal properties of (23) are best described by introducing three abstract spaces I_a, I_b and I_- with the attributions given in Table I. The usual isotopic spin space is then the space orthogonal to the 4-th axis in the 4-space obtained as the direct product of I_a and I_b.

TABLE I. – *Baryons in abstract spaces.*

	I_a	I_b	I_{b3}		I_-	I_{-3}	
N, Ξ	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Y, Z	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0

([12]) Except if more than four vetons are introduced.

One can similarly define a 3-space E using I_b and I_- instead of I_a and I_b. w is then a *scalar* in I_- and a *vector component* in I, wihle r is a *scalar* in I_a and a *vector component* in E. As regards strong interactions, only the invariance in I space is postulated.

6. – Conclusions.

It has been shown that the veton theory can receive at least two qualitatively different formulations. Both of them are in agreement with present experimental knowledge. Both of them lead to predictions which are in some respects different from those of the schizon theory. For one of them (scheme A) these observable differences, which are described in Section **4**, bear on the results of some of the experiments, suggested by LEE and YANG in order to test the validity of the intermediate boson assumption. This first formulation (scheme A), although it is not as « natural » as the schizon theory (cf. Section **3**), is therefore a possible alternative to it, to which the absence of any observed neutral leptonic current effects (proposition iv) in Section **1**) gives some preliminary support and which could be proved or disproved by experiment. The other formulation (scheme B) differs from the schizon theory in that it definitely predicts the $\Delta S = 2$ *leptonic* decay of the Ξ (reactions (16)) which is completely forbidden in the schizon theory as well as in scheme A. If scheme A is correct, these decays should occur with a decay rate proportional not to the square of the β-decay coupling constant but to the square of the, weaker, Λ leptonic decay coupling constant: their branching ratio against normal Ξ-decay should then be ([13]) $1:20$ to $1:50$. Scheme B is on the other hand much more « natural » from a theoretical point of view than scheme A (see Sections **3** and **5**) and, indeed, can be compared with the schizon theory in that respect, with of course the supplementary advantage that it implies proposition iv) of Section **1** on the absence of neutral leptonic current effects. For these reasons an experimental study of the leptonic decay of the Ξ, and a particular search for processes (16), would be of considerable interest for the theory.

<p style="text-align:center">* * *</p>

([13]) We use the « Table of the leptonic decay rates of the hyperon » by Y. YAMAGUCHI: *CERN report* No. 59-18 (1959).

RIASSUNTO (*)

Le regole $|\Delta S| = 1$ per il decadimento non leptonico e $|\Delta I| = \frac{1}{2}$ sono giustificate da una teoria in cui le interazioni deboli sono dovute a bosoni intermedi dotati di carica, che non hanno un corrispondente neutro. I decadimenti leptonici delle particelle strane in cui la carica totale dei leptoni è zero, sono quindi proibiti, d'accordo con gli esperimenti. Si discutono le differenze osservabili con la predizione della teoria di Lee ed Yang. Sebbene il modo di decadimento leptonico con $\Delta S = 2$ delle particelle delle cascate possa anche essere proibito, per una opportuna scelta dei parametri, la teoria dà tuttavia alcuni argomenti a favore della sua esistenza, con una costante di accoppiamento uguale a quella del modo di decadimento leptonico del Λ, cioè con una velocità di decadimento circa venti volte più piccola della velocità del normale decadimento Ξ.

On Some Compound Models for the Strong Interactions.

M. Tonin

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

Summary. — In this work, the possibility that the strongly interacting particles are compound structures of fundamental Majorana fields is investigated from the point of view of their symmetry properties, assuming that the strong interactions possess larger symmetries than the one leading to the baryon number, isotopic spin and strangeness conservation. The problem of the parity conservation in the investigated theories is also discussed.

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1. – The problem of the understanding of the main properties of the strong interactions has been recently examined by Thirring [1,2] from the point of view of determining the minimum number of fundamental fields which are necessary in order to guarantee the conservation of baryonic number, isotopic spin and strangeness. The result of this investigation is that it is necessary to introduce 6 Majorana fields *i.e.* 3 Dirac fields if one requires that the theory which is *CP* invariant by itself, becomes separately *C* and *P* invariant, while if one renounces to this condition three Majorana fields are sufficient. It is quite apparent that this result leads in a certain way authomatically to the Sakata model and, in fact, in the work of Thirring the three fields are identified for their quantum numbers with the p, n, Λ.

But it is quite possible that the strong interactions possess larger symmetries than the symmetries which lead to the baryon number, isotopic spin and strangeness conservation and in fact many authors in the last years

[1] W. Thirring: *Nucl. Phys.*, **10**, 97 (1959).
[2] W. Thirring: *Nucl. Phys.*, **14**, 565 (1960).

have pointed out such a possibility. The starting point of all these attempts is the idea of the splitting of the four hyperon states with zero hypercharge into two doublets according to Gell-Mann's assumptions leading to the so called doublet approximation formulated by TIOMNO ([3]) and DALLAPORTA ([1]) and extensively discussed by PAIS ([5]). Recently in a series of papers ([6-8]) the possibility has been pointed out that the K interactions are invariant with respect to rotations in a four dimensional (hypercharge) space. This invariance leads to the so called hypercharge and hypernumber independence. The hypercharge and the hypernumber are transformations mixing respectively, the hypercharge, the nucleons and the Ξ baryons and the hypernumber, the Σ, Λ hyperons, in the same way in which isotopic spin mixes p-n, Σ^+-Y_0, Z_0-Σ^-, Ξ^0-Ξ^-.

Although these invariances are partly destroyed by the perturbation which splits the masses of the baryons and the effect of which is considered in ([6-8]), the possibility that such extended invariance could be a property of the fundamental interactions should be of importance for their understanding in a more fundamental theory than the present attempts. In this case it can be of some interest to extend Thirring's method to such a symmetry group and to see what is the minimum number of fields necessary to describe the strong interactions which would guarantee not only baryon number, isotopic spin and strangeness conservation, but also hypercharge and hypernumber independence.

2. – According to the previous discussion we shall start from a symmetry group larger than the one examined by THIRRING, i.e. the group containing baryon number, isotopic spin, hypercharge and hypernumber invariance and we shall develop a compound model theory which is invariant under this high symmetry group (which in the following will be denoted by HSG).

We assume that the n fundamental fields of the theory are all Majorana fields and therefore that the bosons are Fermi-Yang type compound structures of them. It is known ([9]) that the free lagrangian and the commutation rules of n Majorana fields are invariant under the n-dimensional unitary group if they have vanishing masses but are only invariant under the real orthogonal

([3]) J. TIOMNO: Nuovo Cimento, **6**, 69 (1957).

([4]) N. DALLAPORTA: Proceedings of the Conference on Mesons and recently discovered particles, **5**, 3 (Padua-Venice, 1957); Nuovo Cimento, **7**, 200 (1958).

([5]) A. PAIS: Phys. Rev., **110**, 574 (1958).

([6]) N. DALLAPORTA and T. TOYODA: Nuovo Cimento, **12**, 539 (1959).

([7]) N. DALLAPORTA and T. TOYODA: Nuovo Cimento, **14**, 142 (1959); N. DALLAPORTA and V. DE SANTIS: Nuovo Cimento, **14**, 225 (1959).

([8]) N. DALLAPORTA and L. K. PANDIT: Nuovo Cimento, **16**, 135 (1960).

([9]) B. TOUSCHEK: Lectures given at the International School of Varenna (1959).

n-dimensional group if the Majorana fields masses are equal but not vanishing. These groups are represented respectively by the transformations

$$(1) \qquad \Psi' = \exp\left[A + i\gamma_5 S\right]\Psi ,$$

$$(2) \qquad \Psi' = \exp\left[A\right]\Psi ,$$

where A is the most general real and antisymmetric $n \times n$ matrix, S is the most general real and symmetric $n \times n$ matrix and Ψ denotes the set of n Majorana fields.

Thus, the problem is to find a n-dimensional representation for the HSG whose representative space is mapped by the n fundamental fields, and which is a subgroup of the transformation group (1). Moreover if we want that the fields have non-vanishing masses, this representation must be contained in the transformation group (2) or in a subgroup of (1) isomorphic to the group (2).

The HSG's generators are the ten hermitian operators representing the baryon number N, the isotopic spin T_i, the hypercharge Y_i and the hypernumber Z_i ($i = 1, 2, 3$), and the Lie algebra of this group is expressed by the following commutations rules

$$(3) \quad [Y_1, Y_2]_- = iY_3 ; \quad [Z_1, Z_2]_- = iZ_3 ; \quad [T_1, T_2] = iT_3 \qquad + cycl ;$$

$$[Y_i, Z_j]_- = [Y_i, T_j]_- = [Z_i, T_j]_- = [N, T_i]_- = [N, Y_i]_- = [N, Z_i]_- = 0$$

Moreover we shall consider the unitary and space-time independent operator R with the commutation rules

$$(4) \qquad [R, Y_i]_- = [R, Z_i]_- = [R, T_i]_- = 0 , \qquad [R, N]_+ = 0 ,$$

R therefore represents the charge conjugation or more precisely the spinor conjugation. This transformation is the product between the charge and the boson conjugation, the last being a particular transformation of the HSG [10].

From the commutation rules (3) we can see that the HSG can be decomposed into the direct product of four groups; three of them are $3D$ rotation groups corresponding respectively to the isotopic spin, hypercharge and hypernumber rotations, and the other is the $2D$ rotation group leading to the baryon number conservation

$$\text{HSG} \quad R_3 \times R_3' \times R_3'' \times R_2 .$$

Therefore all the irreducible representations of the HSG can be obtained as the direct product of irreducible representations of these groups. The required

([10]) P. BUDINI, N. DALLAPORTA and C. FONDA: *Nuovo Cimento*, **9**, 316 (1958).

representation (which should not be necessarely irreducible) must generally be a direct sum of these irreducible representations.

In order to select all the possible representations it is necessary to verify if we can build the real baryons and bosons with their correct quantum numbers (see Table I, ref. ([8])), as compound structures of the representative vectors. In particular the baryons must be built up with an odd number of fundamental fields in order to assure the connection between spin and statistics according to the Pauli principle.

First we do not consider the baryon number and we restrict to the group $R_3 \times R_3' \times R_3''$. If we consider only scalar and spin one half representations of the $3D$ rotation groups, only the following irreducible representations are possible:

	Dimension
$\mathcal{D}_0 \times \mathcal{D}_0' \times \mathcal{D}_0''$	1
$\mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_0' \times \mathcal{D}_0''$	2
$\mathcal{D}_0 \times \mathcal{D}_{\frac{1}{2}}' \times \mathcal{D}_0''$	2
$\mathcal{D}_0 \times \mathcal{D}_0' \times \mathcal{D}_{\frac{1}{2}}''$	2
$\mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_{\frac{1}{2}}' \times \mathcal{D}_0''$	4
$\mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_0' \times \mathcal{D}_{\frac{1}{2}}''$	4
$\mathcal{D}_0 \times \mathcal{D}_{\frac{1}{2}}' \times \mathcal{D}_{\frac{1}{2}}''$	4
$\mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_{\frac{1}{2}}' \times \mathcal{D}_{\frac{1}{2}}''$	8

Since some real baryons belong to the zero representation and some other to the one half representation of the hypercharge and hypernumber, these two possibilities must appear together in the required reducible representation. Moreover, also the one half representation of the isotopic spin must appear. Therefore the simpler possibilities are:

$$
(5) \quad \begin{cases} A) & \mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_0' \times \mathcal{D}_0'' + \mathcal{D}_0 \times \mathcal{D}_{\frac{1}{2}}' \times \mathcal{D}_0'' + \mathcal{D}_0 \times \mathcal{D}_0' \times \mathcal{D}_{\frac{1}{2}}'' \qquad 6 \\ B) & \mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_{\frac{1}{2}}' \times \mathcal{D}_0'' + \mathcal{D}_{\frac{1}{2}} \times \mathcal{D}_0' \times \mathcal{D}_{\frac{1}{2}}'' . \qquad 8 \end{cases}
$$

In order to introduce also the baryon number, we note first that all the irreducible representations of the $2D$ rotation group have dimension one. Now, we consider only the representations $\mathcal{U}^{(0)}$ and $\mathcal{U}^{(1)}$ corresponding respectively to fundamental fields having baryon number 0 or 1.

To obtain from representation (5) a complete representation of the HSG we can proceed in two ways: *i.e.*, we can take the direct product of the repre-

sentations (5) and $\mathscr{U}^{(1)}$ or alternatively we can consider the sum of the direct product of representations (5) by $\mathscr{U}^{(0)}$ and $\mathscr{L}^0 \times \mathscr{L}'_0 \times \mathscr{L}''_0 \times \mathscr{U}^{(1)}$. Now we must get rid of the cases $[\mathscr{D}_{\frac{1}{2}} \times \mathscr{D}'_0 \times \mathscr{D}''_0 + \mathscr{D}_0 \times \mathscr{D}'_{\frac{1}{2}} \times \mathscr{L}''_0 + \mathscr{D}_0 \times \mathscr{D}'_0 \times \mathscr{D}''_{\frac{1}{2}}] \times \mathscr{U}^{(1)}$ and $\mathscr{D}_{\frac{1}{2}} \times [\mathscr{D}'_{\frac{1}{2}} \times \mathscr{D}''_0 + \mathscr{D}'_0 \times \mathscr{D}''_{\frac{1}{2}}] \times \mathscr{U}^{(0)} + \mathscr{D}_0 \times \mathscr{D}'_0 \times \mathscr{D}''_0 \times \mathscr{U}^{(1)}$ because they give us baryons that do not satisfy the correct connection between spin and statistics and therefore the following representations remain

Dimension

$$(6) \quad \begin{cases} A) \quad [\mathscr{D}_{\frac{1}{2}} \times \mathscr{D}'_0 \times \mathscr{D}''_0 + \mathscr{D}_0 \times \mathscr{D}'_{\frac{1}{2}} \times \mathscr{D}''_0 + \mathscr{D}_0 \times \mathscr{D}'_0 \times \mathscr{D}''_{\frac{1}{2}}] \times \mathscr{U}^{(0)} + \\ \qquad\qquad\qquad + \mathscr{D}_0 \times \mathscr{D}'_0 \times \mathscr{D}''_0 \times \mathscr{U}^{(1)} \qquad\qquad 7 \\ \\ B) \quad \mathscr{D}_{\frac{1}{2}} \times [\mathscr{D}'_{\frac{1}{2}} \times \mathscr{D}''_0 + \mathscr{D}'_0 \times \mathscr{D}''_{\frac{1}{2}}] \times \mathscr{U}^{(0)} . \qquad\qquad\qquad 8 \end{cases}$$

So far, we have not yet considered the charge conjugation. Since it anticommutes with the baryon number, it can never be assumed until we consider only one-dimensional representations of the baryon number. In order to introduce charge conjugation we must, therefore, substitute to $\mathscr{U}^{(1)}$ the reducible representation $\mathscr{V}^{(1)} = \begin{pmatrix} \mathscr{U}^{(1)} & 0 \\ 0 & \mathscr{U}^{(-1)} \end{pmatrix}$ so that the representation (6A), becomes:

$$(7A) \quad [\mathscr{D}_{\frac{1}{2}} \times \mathscr{D}'_0 \times \mathscr{D}''_0 + \mathscr{D}_0 \times \mathscr{D}'_{\frac{1}{2}} \times \mathscr{D}''_0 + \mathscr{D}_0 \times \mathscr{D}'_0 \times \mathscr{D}''_{\frac{1}{2}}] \times \mathscr{U}^0 + \mathscr{D}_0 \times \mathscr{D}'_0 \times \mathscr{D}''_0 \times \mathscr{V}^{(1)} :$$

Dimension 8

and the representation (6B) becomes:

$$(7B) \qquad\qquad \mathscr{D}_{\frac{1}{2}} \times [\mathscr{D}'_{\frac{1}{2}} \times \mathscr{D}''_0 + \mathscr{D}'_0 \times \mathscr{D}''_{\frac{1}{2}}] \times \mathscr{V}^{(1)} : \qquad \text{Dimension 16} .$$

3. – It is now easy to write explicitly in the form $A + i\gamma_5 S$ the representations of the ten generators of the HSG for the different cases (6A), (6B), (7A), (7B) previously discussed.

A) 7 (8) *fields theory.* – A 7×7 representation of the elements of the HSG's algebra is (*):

$$(8) \quad \begin{cases} y_i = \Gamma_i \begin{pmatrix} \sigma_i & & \\ & 0 & \\ & & 0 \\ & & & 0 \end{pmatrix} & z_i = \Gamma_i \begin{pmatrix} 0 & & \\ & \sigma_i & \\ & & 0 \\ & & & 0 \end{pmatrix} , \\ \\ t_i = \Gamma_i \begin{pmatrix} 0 & & \\ & 0 & \\ & & \sigma_i \\ & & & 0 \end{pmatrix} & n = \gamma_5 \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \\ & & & 1 \end{pmatrix} , \end{cases}$$

(*) As in ref. (^{1}), we shall indicate with small letters the matrices representing the operators denoted with capital letters.

where σ_i are Pauli matrices, 0 are 2×2 matrices having all their elements equal zero, Γ_i are γ_5 or I according to $i = 1, 3$ or $i = 2$. The matrices (8) are not all antisymmetric and therefore the fundamental fields bare masses must be vanishing. From representation (8) we deduce that if we indicate by

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \varphi \end{pmatrix}$$

the set of the 7 Majorana spinors, where χ_1, χ_2, χ_3 are pairs of Majorana fields carrying respectively hypercharge, hypernumber and isotopic spin, and φ is a Majorana field carrying baryon number, the elementary particles have the following structures:

Baryons: $(p, n, \Xi^0, \Xi^-) = \chi_1\chi_3(1 + \gamma_5)\varphi\,|0\rangle$; $(\Sigma^+ Y_0 Z_0 \Sigma^-) = \chi_2\chi_3(1 + \gamma_5)\varphi\,|0\rangle$

Pions: $\pi_i = \chi_3\sigma_i\chi_3\,|0\rangle$;

K-mesons: $K_\mu = (\chi_1 \chi_2)\Omega_\mu\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}|0\rangle$; $\mu = 1, ..., 4,$

where

$$\Omega_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \Omega_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$$

and the K-mesons are related to the K_μ by the relations:

$$K^\pm = K_1 \pm iK_2 ; \qquad K_0 = K_3 + iK_4 ; \qquad \bar{K}_0 = K_3 - iK_4 .$$

This theory cannot contain the charge conjugation and therefore parity conservation cannot be assured. In order to introduce charge conjugation we must use 8×8 matrices instead of 7×7 matrices by putting $n = \gamma_5 \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \sigma_3 \end{pmatrix}$

and interpreting as 2×2 matrices also the last zeros of matrices (8).

8 (16) *Fields theory.* – A 8×8 representation of the HSG is

(9)
$$\begin{cases} y_i = \Gamma_i(\tau_0 - \tau_3)\omega_i\sigma_0 , & z_i = \Gamma_i(\tau_0 + \tau_3)\omega_i\sigma_0 , \\ t_i = \Gamma_i(\tau_0 - \tau_3)\omega_0\sigma_i , & n = \gamma_5\tau_0\omega_0\sigma_0 , \end{cases}$$

where Γ_i are γ_5 or I according to $i = 1, 3$ or $i = 2$, τ_0, ω_0, σ_0 are 2×2 unit matrices, τ_i, ω_i, σ_i are Pauli matrices and the matrices (9) (without considering γ_5) must be thought as 8×8 matrices obtained carrying out the direct product of three 2×2 matrices. The matrices (9) are contained in the group of 8×8 matrices $A + i\gamma_5 S$ and therefore, they give us an 8-dimensional representation of the HSG. But these matrices are not all antisymmetric and it is impossible to find out a unitary transformation able to give them this property. It follows that in a 8 fields theory the fundamental fields must have vanishing bare mass. If we indicate with ψ the set of 8 Majorana fields we can easily find the following structure for the elementary particles

$$(\text{baryons}) = (1 + \gamma_5)\psi\,|0\rangle\,, \qquad (\text{antibaryons}) = (1 - \gamma_5)\psi\,|0\rangle\,,$$

$$(\text{pion}): \quad \pi_i = \begin{cases} \psi^\dagger(1 + \gamma_5)\tau_0\omega_0\sigma_i(1 + \gamma_5)\psi\,|0\rangle\,, \\ \text{or} \\ \psi^\dagger(1 + \gamma_5)\tau_3\omega_0\sigma_i(1 + \gamma_5)\psi\,|0\rangle\,, \end{cases}$$

$$(\text{K-mesons}): \quad K_\mu = \begin{cases} \psi^\dagger(1 + \gamma_5)\Omega_\mu(1 + \gamma_5)\psi\,|0\rangle\,, \\ \text{or} \\ \psi^\dagger(1 + \gamma_5)\Omega_\mu\Omega_5(1 + \gamma_5)\psi\,|0\rangle\,, \end{cases}$$

where

$$\Omega_4 = \tau_1\omega_0\sigma_0\,; \qquad \Omega_i = \tau_2\omega_i\sigma_0\,.$$

It is clear from the expression of n that it is impossible to find out a unitary space time independent matrix, anticommuting with n and therefore, representing the charge conjugation. To obtain it we must consider a 16-dimensional representation, for instance:

$$(10) \quad \begin{cases} \begin{cases} y_1 = \gamma_5\varrho_3(\tau_0 - \tau_3)\omega_1\sigma_0 \\ y_2 = \varrho_0(\tau_0 - \tau_3)\omega_2\sigma_0 \\ y_3 = \gamma_5\varrho_3(\tau_0 - \tau_3)\omega_3\sigma_0 \end{cases} \\ \begin{cases} t_1 = \gamma_5\varrho_3\tau_0\omega_0\sigma_1 \\ t_2 = \varrho_0\tau_0\omega_0\sigma_2 \\ t_3 = \gamma_5\varrho_3\tau_0\omega_0\sigma_3 \end{cases} \end{cases} \qquad \begin{cases} z_1 = \gamma_5\varrho_3(\tau_0 + \tau_3)\omega_1\sigma_0 \\ z_2 = \varrho_0(\tau_0 + \tau_3)\omega_2\sigma_0 \\ z_3 = \gamma_5\varrho_3(\tau_0 + \tau_3)\omega_3\sigma_0 \end{cases}$$

$$n = \gamma_5\varrho_3\tau_0\omega_0\sigma_0\,.$$

In this case R can be expressed by

$$(10') \qquad\qquad\qquad r = \varrho_2\omega_2\sigma_2\,.$$

Obviously in (10) and (10′) ϱ_i are Pauli matrices and ϱ_0 is a 2×2 unit matrix. If we apply to representation (10) the unitary and real transformation

$$S = \frac{1}{\sqrt{2}}(1 - i\gamma_5 \varrho_1) ; \qquad S^* = S , \quad S^\dagger = S^{-1} ,$$

we obtain

(11)

$$y_1 = \varrho_2(\tau_0 - \tau_3)\omega_1\sigma_0 \qquad\qquad z_1 = \varrho_2(\tau_0 + \tau_3)\omega_1\sigma_0$$

$$y_2 = \varrho_0(\tau_0 - \tau_3)\omega_2\sigma_0 \qquad\qquad z_2 = \varrho_0(\tau_0 + \tau_3)\omega_2\sigma_0$$

$$y_3 = \varrho_2(\tau_0 - \tau_3)\omega_3\sigma_0 \qquad\qquad z_3 = \varrho_2(\tau_0 + \tau_3)\omega_3\sigma_0$$

$$t_1 = \varrho_2\tau_0\omega_0\sigma_1$$

$$t_2 = \varrho_0\tau_0\omega_0\sigma_2 \qquad\qquad n = \varrho_2\tau_0\omega_0\sigma_0 .$$

$$t_3 = \varrho_2\tau_0\omega_0\sigma_3$$

Since the matrices (11) are all real and antisymmetric, it follows that in the 16 fields case we can assure the HSG invariance also with non vanishing mass fields. Starting from (11) and using the transformation $\mathscr{U} = (1/\sqrt{2})(1 + i\varrho_1)$ we obtain a representation where $n = \varrho_3\tau_0\omega_0\sigma_0$ and, therefore, where the first (8) components of ψ are creation operators of particles and the others, creation operators of antiparticles. In this case the HSG generators can be so represented:

(12)

$$y_1 = \varrho_3(\tau_0 - \tau_3)\omega_1\sigma_0 \qquad\qquad z_1 = \varrho_3(\tau_0 + \tau_3)\omega_1\sigma_0$$

$$y_2 = \varrho_0(\tau_0 - \tau_3)\omega_2\sigma_0 \qquad\qquad z_2 = \varrho_0(\tau_0 + \tau_3)\omega_2\sigma_0$$

$$y_3 = \varrho_3(\tau_0 - \tau_3)\omega_3\sigma_0 \qquad\qquad z_3 = \varrho_3(\tau_0 + \tau_3)\omega_3\sigma_0$$

$$t_1 = \varrho_3\tau_0\omega_0\sigma_1$$

$$t_2 = \varrho_0\tau_0\omega_0\sigma_2 \qquad\qquad n = \varrho_3\tau_0\omega_0\sigma_0 .$$

$$t_3 = \varrho_3\tau_0\omega_0\sigma_3$$

From (12) we can see that the 16 components of ψ have the following physical interpretation

$$\psi = \begin{pmatrix} \varphi \\ i\varphi^c \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} p \\ n \\ \Xi^0 \\ \Xi^- \\ \Sigma^+ \\ Y^0 \\ Z_0 \\ \Sigma^- \end{pmatrix}$$

with an obvious meaning for the used symbols.

Obviously the representation (12) is only one of the many possible representations and has the advantage that the components of ψ in this representation have a clear physical meaning. In some cases other representations can appear more useful. These can be obviously obtained from (12) with some convenient unitary transformations and, as an example, we list a few cases which have been already considered in the literature.

1) Using the transformation

$$A(\sigma, \varrho) = \tfrac{1}{2}[\alpha\sigma_1 + \beta\sigma_3][(1 + \sigma_3) + \varrho_1(1 - \sigma_3)] \quad \text{where} \quad \begin{aligned} \alpha &= 1 - \gamma_5\varrho_3 \\ \beta &= 1 + \gamma_5\varrho_3 \end{aligned}$$

we obtain the Gürsey [11] representation having the property that the baryon number is represented by $\psi' = \exp[i\alpha\gamma_5]\psi$ and the isobaric spin by

(13) $$\psi' = \alpha\psi + b\gamma_5\psi^\circ \qquad \text{with } |a|^2 + |b|^2 = 1$$

We have

$$\begin{cases} y_1 = \gamma_5\varrho_0(\tau_0 - \tau_3)\omega_1\sigma_3 \\ y_2 = \varrho_0(\tau_0 - \tau_3)\omega_2\sigma_0 \\ y_3 = \gamma_5\varrho_0(\tau_0 - \tau_3)\omega_3\sigma_3 \end{cases} \qquad \begin{array}{l} z_1 = \gamma_5\varrho_0(\tau_0 + \tau_3)\omega_1\sigma_3 \\ z_2 = \varrho_0(\tau_0 + \tau_3)\omega_2\sigma_0 \\ z_3 = \gamma_5\varrho_0(\tau_0 + \tau_3)\omega_3\sigma_3 \end{array}$$

$$\begin{cases} t_1 = \gamma_5\varrho_1\tau_0\omega_0\sigma_0 \\ t_2 = \gamma_5\varrho_2\tau_0\omega_0\sigma_0 \\ t_3 = \varrho_3\tau_0\omega_0\sigma_0 \end{cases} \qquad n = \gamma_5\varrho_0\tau_0\omega_0\sigma_3 \,.$$

2) With the transformation $A(\omega, \varrho)$ we obtain the Dallaporta-Toyoda [6] representation where the hypercharge and the hypernumber have the expression (13):

$$\begin{cases} y_1 = \gamma_5\varrho_1(\tau_0 - \tau_3)\omega_0\sigma_0 \\ y_2 = \gamma_5\varrho_2(\tau_0 - \tau_3)\omega_0\sigma_0 \\ y_3 = \varrho_3(\tau_0 - \tau_3)\omega_0\sigma_0 \end{cases} \qquad \begin{cases} z_1 = \gamma_5\varrho_1(\tau_0 + \tau_3)\omega_0\sigma_0 \\ z_2 = \gamma_5\varrho_2(\tau_0 + \tau_3)\omega_0\sigma_0 \\ z_2 = \varrho_3(\tau_0 + \tau_3)\omega_0\sigma_0 \end{cases}$$

$$\begin{cases} t_1 = \gamma_5\varrho_0\tau_0\omega_3\sigma_1 \\ t_2 = \varrho_0\tau_0\omega_0\sigma_2 \\ t_3 = \gamma_5\varrho_0\tau_0\omega_0\sigma_3 \end{cases} \qquad n = \gamma_5\varrho_0\tau_0\omega_3\sigma_0 \,.$$

[11] F. Gürsey: *Nuovo Cimento*, **7**, 411 (1958).

3) At last we obtain the completely mixed Dallaporta-Toyoda [7] representation by putting

$$B(\sigma, \varrho_3) = \frac{1}{\sqrt{2}} (\alpha\sigma_1 + \beta\sigma_3)$$

and using the transformation

$$B(\omega, \varrho_3) \cdot B(\sigma, \varrho_3) .$$

Then

$$
\begin{cases}
y_1 = - \gamma_5 \varrho_0 (\tau_0 - \tau_3) \omega_1 \sigma_0 \\
y_2 = \varrho_0 (\tau_0 - \tau_3) \omega_2 \sigma_0 \\
y_3 = \gamma_5 \varrho_0 (\tau_0 - \tau_3) \omega_3 \sigma_0
\end{cases}
\qquad
\begin{cases}
z_1 = - \gamma_5 \varrho_0 (\tau_0 + \tau_3) \omega_1 \sigma_0 \\
z_2 = \varrho_0 (\tau_0 + \tau_3) \omega_2 \sigma_0 \\
z_3 = \gamma_5 \varrho_0 (\tau_0 + \tau_3) \omega_3 \sigma_0
\end{cases}
$$

$$
\begin{cases}
t_1 = - \gamma_5 \varrho_0 \tau_0 \omega_0 \sigma_1 \\
t_2 = \varrho_0 \tau_0 \omega_0 \sigma_2 \\
t_3 = \gamma_5 \varrho_0 \tau_0 \omega_0 \sigma_3
\end{cases}
\qquad
n = \gamma_5 \varrho_0 \tau_0 \omega_0 \sigma_3
$$

4. The non invariance of type A theories with 7 fields and of type B theories with 8 fields, under charge conjugation and therefore under P inversion, does not necessarily imply that one must at once disregard these theories. In fact, as THIRRING has shown in [1,2] for his three fields theory, the contained symmetries are sufficient to assure that in some boson vertices (also renormalized) the parity is nevertheless conserved. This property is valid only for the pion vertices in the case of the Thirring model, but in our case it is valid also for the K vertices if only we extend a little the HSG.

For a better understanding of this problem, let us examine the P conservation considering only the case of the 8 fields, B type theory which from a physical point of view, seems to us more attractive than the other.

Now, if the 8 fermion masses are equal, the boson renormalized vertices must generally by expressed by [2]:

(14) $$\langle p_+|O_i|p_+\rangle = \chi^*(p')A_i\gamma_4\chi(p) + \chi^*(p')B_i\gamma_4\gamma_5\chi(p) ,$$

where A_i and B_i are 8×8 matrices, $i = 1, ..., n$, n being the number of bosons belonging to the same multiplet and χ is the set of 8 wave functions which are solutions of the mass m Dirac equation. The hermiticity and the time reversal invariance of the theory involve [2]:

(15) $$A^T = A , \qquad B^T = - B ,$$

or

(15′) $$A^T = -A , \qquad B^T = B .$$

Moreover the HSG invariance for pion vertices requires that A_i and B_i must have the following structure:

(16)
$$\begin{cases} A_i = a_1 \tau_0 \omega_0 \sigma_i + a_2 \tau_3 \omega_0 \sigma_i , \\ B_i = b_1 \tau_0 \omega_0 \sigma_i + b_2 \tau_3 \omega_0 \sigma_i , \end{cases}$$

where $a_{1,2}$ and $b_{1,2}$ are real or pure imaginary numbers. Now (16) have a well defined symmetry character (for instance $A_1^T = A_1$ and $B_1^T = B_1$) and therefore the relations (15) or (15′) can be satisfied only if $A = 0$ or $B = 0$. It follows that for the pion vertices we have P conservation.

However for the K vertices, the HSG invariance, as it stands, is not sufficient to assure P conservation for two reasons:

1) It does not guarantee that the p, n, Ξ^0, Ξ^- and the Σ, Λ hyperons masses are equal and therefore, the equation (14) does no longer hold.

2) Even if we suppose that the masses of all the 8 baryons are equal neglecting perturbations, the HSG invariance requires for the matrices A and B the following expression

(17)
$$\begin{cases} A_\mu = a_1 \Omega_\mu + a_2 \Omega_\mu \Omega_5 , \\ B_\mu = b_1 \Omega_\mu + a_2 \Omega_\mu \Omega_5 . \end{cases} \qquad (\mu = 1, ..., 4),$$

Now Ω_μ and $\Omega_\mu \Omega_5$ have opposite symmetry character (for instance $A_1^T = -a_1 \Omega_1 + a_2 \Omega_1 \Omega_5$, $B_1^T = -b_1 \Omega_1 + b_2 \Omega_1 \Omega_5$) and therefore in this case the conditions (15) and (15′) can be verified also if A_μ and B_μ are both non vanishing, assuming A_μ to be proportional to Ω_μ and B_μ proportional to $\Omega_\mu \Omega_5$ or viceversa. But if we suppose that the theory is also invariant under the unitary transformation

(18) $$\psi' = \Omega_4 \psi ,$$

we can assure P conservation also for K vertices. In fact in this case we can immediately infer that

1) the p, n, Ξ^0, Ξ^- and Σ, Λ hyperon masses are equal because (18) is the transformation connecting these two sets of particles;

2) the second term in (17) must be disregarded in order to assure invariance of (14) under (18), because (18) represents the fourth axis inversion in the hypercharge and hypernumber space.

The result that the P invariance cannot be introduced in the theory as a rigorous invariance, implies however that in the strong interactions there must be a P non conservation source that may be searched for, for instance, in a « strong » Fermi interaction between four baryons. Anyway, we can conjecture that the effect of such an interaction will be important only at high energies and that at lower energies the interaction with boson exchange prevails because this latter has a longer range. Therefore, the 8 fields theory is not in contradiction with the experiments of nuclear physics and with the recent experiments on the strange particles indicating that P is conserved in strong interactions at least at low energies.

<center>* * *</center>

<center>RIASSUNTO</center>

In questo lavoro la possibilità che le particelle interagenti fortemente siano strutture composte di campi fondamentali di Majorana, viene discussa dal punto di vista delle proprietà di simmetria delle interazioni forti nell'ipotesi che queste ultime possiedano simmetrie più vaste di quelle che danno luogo alla conservazione del numero barionico, dello spin isotopico e della stranezza. Inoltre si discute il problema della conservazione della parità per le teorie prese in esame.

The Field Theoretic Definition of the Nuclear Potential (*) - III.

J. M. CHARAP (**) and M. J. TAUSNER (**)

The Enrico Fermi Institute for Nuclear Studies and the Department of Physics,
The University of Chicago - Chicago. Ill.

Summary. — The scattering of nucleons at energies below the meson production threshold is discussed; the amplitude is a sum of five terms, each proportional to a different spin invariant. A comparison in perturbation theory, between the field-theoretic amplitude and that from potential theory enables one to conclude that a potential which will reproduce the field-theoretic amplitude through the two-meson exchange contributions exists. This potential is also a sum of five terms and is « local » in the sense that for a suitable choice of the spin-dependent operators, their coefficients are energy-independent. The potential is derived, apart from radiative and rescattering corrections; it is again observed that the static limit $(\mu/M \to 0)$ is not well defined. The spectral functions for the potential are calculated. Rescattering corrections, etc., are discussed and, assuming simple analyticity properties for Feynman graphs, are shown not to destroy the locality of the potential. It is not yet possible to furnish a proof independent of perturbation theory that a potential exists, as was done for spinless particles in Part I, because the necessary dispersion relations have not been proved; in fact it is not even known for which choice of spin invariants they might be expected to hold. Some clues as to this choice of « proper » spin invariants are afforded by the limitations imposed in order that the fourth-order potential be energy-independent.

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1. – Introduction.

The object of this series of papers (¹) is to formulate a definition of a nucleon-nucleon potential which, when inserted into a Schrödinger equation, will generate a scattering amplitude which is a good approximation to the field theoretic one at energies below the threshold for meson production. We further require that this potential be a function of only one variable; *i.e.* in momentum space it should depend only on the momentum transfer and not on the energy (this means that the potential is local in configuration space). We impose this restriction because the scattering amplitude itself is a function of momentum transfer and energy and to have produced a «potential» depending on *both* of these variables is to have achieved no great simplification.

It was formerly supposed (²) that the inclusion of nucleon recoil would lead to an energy-dependent potential, and to avoid this the «static approximation» was invoked; *i.e.* an expansion in powers of μ/M was made and only the leading terms were retained. Recent work on the two-nucleon problem (³) has shown quite clearly that the static limit is *not* valid; fortunately it is also not necessary. At energies below the meson production threshold it is possible to define a potential satisfying our conditions which takes recoil effects into account without this approximation. Naturally, both μ and M appear as parameters in this potential. The only approximation made is to consider the initial and final nucleon energies to be small; this restriction is not imposed on the energies of the virtual particles in intermediate states.

In I it was shown that a potential satisfying these conditions exists for the case of a spinless «neutron» and «proton» interacting through a scalar meson field. This was based on the formal identity below the meson production threshold of the unitarity condition and the once-subtracted one-dimensional dispersion relation in field theory (⁴), and in non-relativistic potential theory (⁵). It was further shown possible to give an explicit construction for this potential utilizing the proven Mandelstam representation for the potential theory scattering amplitude (⁶).

(¹) The earlier papers in this series are: J. M. CHARAP and S. P. FUBINI: *Nuovo Cimento*, **14**, 540 (1959) and **15**, 73 (1960), referred to as I and II respectively. This paper however is intended to be self-contained.

(²) Cf. the review articles in *Suppl. Prog. Theor. Phys.*, No. 3 (1596).

(³) M. L. GOLDBERGER and R. OEHME: *Ann. of Phys* (to be published).

(⁴) M. L. GOLDBERGER, Y. NAMBU and R. OEHME: *Ann. of Phys.*, **2**, 226 (1957). (Referred to as GNO).

(⁵) N. N. KHURI: *Phys. Rev.*, **107**, 1148 (1957).

(⁶) R. BLANKENBECLER, M. L. GOLDBERGER, N. N. KHURI and S. B. TREIMAN: *Ann. of Phys.* (to be published); T. REGGE: *Nuovo Cimento*, **14**, 951 (1959); A. KLEIN: *Journ. Math. Phys.*, **1**, 41 (1960).

In II the proof was extended to include charged mesons and the concomitant exchange forces. This involved the construction of an isotopic spin-dependent potential for the scattering of two *distinguishable* nucleons with a subsequent symmetrization.

In this paper we shall consider the effect of the nucleon spin. Through fourth-order perturbation theory, a very naive approach leads to a definition of the potential which satisfies our conditions. This approach is based on a term-by-term identification of the perturbation expansions of the scattering matrices in field theory and in potential theory. Furthermore, in I, it was shown that the potential thus defined agreed with that obtained via the more sophisticated approach using dispersion relations. We are unable to make use of dispersion relations when the nucleons have spin since there exists as yet no proof even of an analogue of a Khuri relation for the most general spin-dependent potential, let alone a proof of a Mandelstam representation. Although we believe that such relations will eventually be proven, we do not here postulate and use them. This is because the scattering matrix, when spins are present, is to be expressed in terms of five amplitudes [7] (for *each* of which we would have to postulate a Mandelstam representation) corresponding to five linearly independent spin invariants. The choice of invariants is *a priori* not unique; it is unlikely that a Mandelstam representation is valid for an arbitrary choice, so until such relations are proved we do not know what is a « proper » choice. We are thus forced to utilize the perturbation theory approach, believing however that the results are independent of the validity of perturbation theory. It will be shown that the requirement that the potentials corresponding to each amplitude depend only on the momentum transfer imposes strong limitations on the choice of invariants. This may give a clue as to the « proper » choice of invariants.

In Section 2 the kinematics and the separation into the five amplitudes are discussed and the potential is defined. Sections 3 and 4 deal, respectively, with the calculations of the field-theoretic amplitudes and the iteration of the Yukawa potential. The cancellation between these, leading to an energy-independent fourth-order potential, is discussed in Section 5, while in Section 6 the spectral representation of this potential is given. The results are summarized and analysed in Section 7.

2. – Definition of the potential.

The notation used will be as in the previous parts of this work except for the modifications required by the introduction of the spin variables. We will

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[7] L. PUZIKOV, R. RYNDIN and YA. SMORODINSKI: *Soviet Physics J.E.T.P*, **5**, 489 (1957); cf. also GNO.

be dealing with the scattering of two particles which initially have four-momenta p_1, n_1 and finally p_2, n_2 (Fig. 1). The center-of-mass three-momentum is $\boldsymbol{k}_1$ initially and $\boldsymbol{k}_2$ finally, with $|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = \eta$. The scattering angle θ is the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. We further define

$$(2.1) \quad \begin{cases} \boldsymbol{\Delta} = \tfrac{1}{2}(\boldsymbol{k}_2 - \boldsymbol{k}_1)\,, & \Delta^2 = \tfrac{1}{2}\eta^2(1 + \cos\theta)\,, \\[2mm] \boldsymbol{K} = \tfrac{1}{2}(\boldsymbol{k}_2 + \boldsymbol{k}_1)\,, & K^2 = \tfrac{1}{2}\eta^2(1 - \cos\theta)\,, \\[2mm] \boldsymbol{n} = \boldsymbol{K}\times\boldsymbol{\Delta}\,, & n^2 = K^2\Delta^2\,, \\[2mm] \multicolumn{2}{c}{E^2 = M^2 + \eta^2\,.} \end{cases}$$
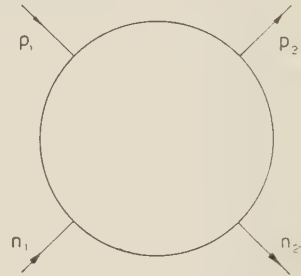

Fig. 1. - Diagram of scattering process.

In accordance with the prescription for handling exchange potentials which has been given in II, we will calculate the potential which reproduces the scattering of two *distinguishable* nucleons, each with spin and isospin one-half. The scattering matrix for the physical case of indistinguishable nucleons is then obtained by antisymmetrizating the scattering matrix resulting from this potential for non identical particles. We will, as usual, suppress the nucleon spin and isospin indices. Thus the potential V, the s-matrix, and the T-matrix will all be operators in the spin and isospin spaces of the nucleons. These are related by

$$(2.2) \qquad (\boldsymbol{k}_2|\,s\,|\boldsymbol{k}_1) = \delta^{(3)}(\boldsymbol{k}_2 - \boldsymbol{k}_1) - 2\pi i\,\delta(W_2 - W_1)(\boldsymbol{k}_2\,|\,T\,|\boldsymbol{k}_1)\,,$$

$$(2.3) \qquad (\boldsymbol{k}_2|\,T\,|\boldsymbol{k}_1) = (\boldsymbol{k}_2|\,V\,|\boldsymbol{k}_1) - \int\!\mathrm{d}^3\boldsymbol{k}\,(\boldsymbol{k}_2|\,V\,|\boldsymbol{k})\,\frac{1}{W - W_1}\,(\boldsymbol{k}|\,T\,|\boldsymbol{k}_1)\,,$$

where in (2.3) the integral over $\boldsymbol{k}$ is to also include a summation over intermediate spin and isospin states, and W is the non-relativistic kinetic energy $|\boldsymbol{k}|^2/M$. (2.3) is of course the familiar Lippman-Schwinger equation.

In field theory the scattering matrix, S, is related to the covariant Feynman amplitude G by

$$(2.4) \qquad (p_2 n_2|\,S\,|p_1 n_1) =$$

$$= \delta^{(3)}(\boldsymbol{p}_2 - \boldsymbol{p}_1)\,\delta^{(3)}(\boldsymbol{n}_2 - \boldsymbol{n}_1) - 2\pi i\,\delta^{(4)}(p_2 + n_2 - p_1 - n_1)\,\frac{M^2}{E^2}\,(\boldsymbol{k}_2|\,G\,|\boldsymbol{k}_1)\,.$$

This is again to be regarded as an operator equation in spin and isospin space.

The symmetries of the two-nucleon system, together with the conservation laws satisfied by the interaction, allow the most general spin-dependent amplitude (for elastic scattering) to be written as a linear combination of five in-

dependent invariants O_i:

$$(2.5) \qquad (\boldsymbol{k}_2 \,|\, T \,|\, \boldsymbol{k}_1) = \sum_{i=1}^{5} O_i T_i(\eta^2, \Delta^2) \,,$$

$$(2.6) \qquad (\boldsymbol{k}_2 \,|\, G \,|\, \boldsymbol{k}_1) = \sum_{i=1}^{5} O_i G_i(\eta^2, \Delta^2) \,.$$

The scalar functions T_i and G_i are still operators in isospin space but are spin independent; they are functions of two independent scalars which we have chosen to be Δ^2 and η^2. We can clearly expect the potential also to be the sum of five terms, each corresponding to a different spin invariant O_i. It is at this point that we run into the first essential difficulty introduced by the presence of spins.

Our basic viewpoint has been to regard the potential as a convenient way of summarizing a very important physical property; *viz.*, that the unitarity condition, together with the once subtracted dispersion relation, determines the scattering matrix, at energies below the meson production threshold, in terms of the zero-energy scattering matrix; *i.e.* the essentially *two*-variable function, the scattering matrix, contains no more information than the *one*-variable function, the zero-energy scattering matrix; or, since we are more used to the Schrödinger equation than to dispersion relations, the potential. It seems pointless to us to deduce a two-variable potential; *i.e.* an energy-dependent or non-local one, since this is just as complicated as the scattering matrix it is meant to simplify. We are prepared to relax this requirement a trifle to allow for the trivial sort of non-locality introduced by, for example, a spin-orbit force. Thus, if we write

$$(2.7) \qquad (\boldsymbol{k}_2 \,|\, V \,|\, \boldsymbol{k}_1) = \sum_{i=1}^{5} O_i V_i \,,$$

we will insist that

$$(2.8) \qquad V_i = V_i(\Delta^2)$$

and that V_i has no dependence on η^2.

The choice of invariants which we have found convenient is

$$(2.9) \qquad \left\{ \begin{array}{l} O_1 = 1 \,, \\[4pt] O_2 = \tfrac{1}{2}\{\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_2 \, \boldsymbol{\sigma}_p \cdot \boldsymbol{k}_1 + \boldsymbol{\sigma}_n \cdot \boldsymbol{k}_2 \, \boldsymbol{\sigma}_n \cdot \boldsymbol{k}_1\} = K^2 - \Delta^2 - i(\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p) \cdot \boldsymbol{n} \,, \\[4pt] O_3 = \boldsymbol{\sigma}_p \cdot \boldsymbol{n} \, \boldsymbol{\sigma}_n \cdot \boldsymbol{n} \,, \\[4pt] O_4 = \boldsymbol{\sigma}_p \cdot \boldsymbol{\Delta} \, \boldsymbol{\sigma}_n \cdot \boldsymbol{\Delta} \,, \\[4pt] O_5 = \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_n \end{array} \right.$$

The O_i are invariant combinations of the spins and momenta, albeit their dependence on the momenta is extremely simple. To form these O_i or any other set of five linearly independent invariants, it is necessary to use $\boldsymbol{K}$ as well as $\boldsymbol{\Delta}$; this is the only source of non-locality which we shall allow. Although our choice of the O_i is not unique, the choice is severely restricted by the requirement that the associated V_i satisfy (2.8).

The perturbation-theoretic definition of the potential is as follows: Consider the perturbation expansion of the Lippman-Schwinger equation (2.3):

$$(2.10) \qquad T = T^{(1)} + T^{(2)} + \dots,$$

$$(2.11) \qquad \begin{cases} (\boldsymbol{k}_2|T^{(1)}|\boldsymbol{k}_1) = (\boldsymbol{k}_2|V^{(1)}|\boldsymbol{k}_1), \\ (\boldsymbol{k}_2|T^{(2)}|\boldsymbol{k}_1) - (\boldsymbol{k}_2|V^{(2)}|\boldsymbol{k}_1) - \int\mathrm{d}^3k\, \dfrac{(\boldsymbol{k}_2|V^{(1)}|\boldsymbol{k})(\boldsymbol{k}|V^{(1)}|\boldsymbol{k}_1)}{W - W_1}. \end{cases}$$

A perturbative expansion of the field-theoretic amplitude yields

$$(2.12) \qquad G = G^{(1)} + G^{(2)} + \dots,$$

where $G^{(1)}$ is the contribution of the one-meson-exchange graph (Fig. 2) and

$$(2.13) \qquad G^{(2)} = G^{(A)} + G^{(B)},$$

Fig. 2. – One-meson exchange graph.

these being the contributions of the two-meson-exchange graphs (Fig. 3). The other fourth-order graphs are taken care of automatically by using renormalized masses and coupling constants.

Identifying T with G order by order, equations (2.10) and (2.11) enable us to define the potential through fourth order as

$$(2.14) \qquad V^{(1)} = T^{(1)} = G^{(1)},$$

$$(2.15) \qquad V^{(2)} = T^{(2)} + I,$$

$$\qquad\qquad - G^{(A)} + G^{(B)} + I,$$

Fig. 3a, 3b. – Two-meson exchange graphs.

where I is the iteration of the Yukawa potential $V^{(1)}$. The $V_i^{(2)}$ defined through (2.15) will be shown to be functions of $\varDelta^2$ alone, in the adiabatic limit for the set of invariants (2.9). By the adiabatic limit we mean $\eta^2 \ll M^2$; this means that the external nucleons are taken to be non-relativistic but there is no restriction on the momenta in the intermediate virtual states. We shall show
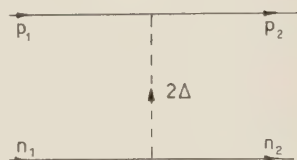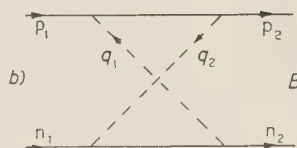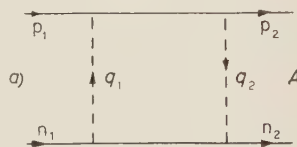
that the recoil contributions from these intermediate states, which are neglected in the more usual static limit ($\mu \ll M$) are a significant part of the potential.

3. – The field-theoretic amplitude.

The field-theoretic amplitude arising from the simple one-meson exchange graph (Fig. 1) is

$$G^{(1)} = \frac{g^2}{(2\pi)^3} \frac{E^2}{M^2} \Lambda^{(+)}(p_2)\gamma_4\gamma_5 \Lambda^{(+)}(p_1) \, \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n \frac{1}{\mu^2 + 4\Delta^2} \Lambda^{(+)}(n_2)\gamma_4\gamma_5 \Lambda^{(+)}(n_1),$$

where $\Lambda^{(+)}(p)$ is the positive-energy projection operator for a nucleon with momentum p and is given by

$$\Lambda^{(+)}(p) = \frac{\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta M + E}{2E},$$

$$(3.1) \qquad G^{(1)} = - \frac{g^2}{(2\pi)^3} \frac{1}{M^2} \, \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n \frac{1}{\mu^2 + 4\Delta^2} (\boldsymbol{\sigma}_p \cdot \boldsymbol{\Delta})(\boldsymbol{\sigma}_n \cdot \boldsymbol{\Delta}),$$

$$(3.2) \qquad \begin{cases} G_4^{(1)} = \dfrac{-g^2}{(2\pi)^3} \dfrac{1}{M^2} \, \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n \dfrac{1}{\mu^2 + 4\Delta^2} , \\[2mm] G_i^{(1)} = 0_i, \, i = 1, 2, 3, 5 . \end{cases}$$

Thus the one-meson exchange, or Yukawa, potential as defined by (2.14) is

$$(3.3) \qquad \begin{cases} V_4^{(1)} = - \dfrac{g^2}{(2\pi)^3} \dfrac{1}{M^2} \, \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n \dfrac{1}{\mu^2 + 4\Delta^2} , \\[2mm] V_i^{(1)} = 0 , \qquad i = 1, 2, 3, 5, \end{cases}$$

and is clearly a function only of Δ^2, independent of η^2.

The contribution $G^{(4)}$ of the uncrossed two-meson exchange graph (Fig. 3a) is

$$(3.4) \quad G^{(4)} = \frac{g^4}{(2\pi)^7} \frac{E^2}{M^2} \Lambda^{(+)}(p_2)\gamma_4\gamma_\mu \Lambda^{(+)}(p_1)(3 - 2\boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n) I_{\mu\nu} \Lambda^{(+)}(n_2)\gamma_4\gamma_\nu \Lambda^{(+)}(n_1) ,$$

where: $I_{\mu\nu} = i \displaystyle\int \mathrm{d}^4Q \, \frac{Q_\mu Q_\nu}{abcd}$,

$$a = Q^2 + 2Q \cdot \Delta + \Delta^2 + \mu^2 - i\varepsilon ,$$

$$b = Q^2 - 2Q \cdot \Delta + \Delta^2 + \mu^2 - i\varepsilon ,$$

$$c = Q^2 + 2Q \cdot P - \Delta^2 - i\varepsilon ,$$

$$d = Q^2 - 2Q \cdot N - \Delta^2 - i\varepsilon ,$$

$$P = \tfrac{1}{2}(p_1 + p_2) , \qquad N = \tfrac{1}{2}(n_1 + n_2) .$$

In Appendix A we show how this may be reduced to the form

$$(3.5) \qquad G'^{(A)} = \frac{g^4}{(2\pi)^7}\,(3 - 2\,\boldsymbol{\tau}_p\!\cdot\!\boldsymbol{\tau}_n)\,\frac{1}{M^2}\sum_{i=1}^{5} O_i g_i^{(A)}\,,$$

$$(3.6) \qquad
\begin{cases}
g_1^{(A)} = \varDelta^2(X + K^2 Y) + \dfrac{1}{4(E + M)^2}\left\{(E + M)^4 - (K^2 - \varDelta^2)^2\right\} F\,,\\[2.2ex]
g_2^{(A)} = (X + K^2 Y) + \dfrac{1}{2(E + M)^2}\left\{(E + M)^2 + K^2 - \varDelta^2\right\} F\,,\\[2.2ex]
g_3^{(A)} = -\,Y - \dfrac{1}{(E + M)^2}\,F\,,\\[2.2ex]
g_4^{(A)} = X\,,\\[1.5ex]
g_5^{(A)} = -\,\varDelta^2 X\,,
\end{cases}$$

$$(3.7) \qquad
\begin{cases}
X = \dfrac{-\pi^2}{8}\displaystyle\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\int_{0}^{1}\mathrm{d}w\, w(1 - w)\left[\mu^2 w + \varDelta^2 w^2(1 - x^2) - \eta^2(1 - w)^2 + \right.\\[2.2ex]
\hspace{6cm}\left. + E^2 y^2(1 - w)^2 - i\varepsilon\right]^{-1}\,,\\[2.6ex]
Y = \dfrac{-\pi^2}{4}\displaystyle\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\int_{0}^{1}\mathrm{d}w\, w(1 - w)^3\left[\mu^2 w + \varDelta^2 w^2(1 - x^2) - \eta^2(1 - w)^2 + \right.\\[2.2ex]
\hspace{6cm}\left. + E^2 y^2(1 - w)^2 - i\varepsilon\right]^{-2}\,,\\[2.6ex]
F = \dfrac{-\pi^2}{4}\displaystyle\int_{-1}^{1}\mathrm{d}x\int_{0}^{1}\mathrm{d}w\, w(1 - w)\left[\mu^2 w + \varDelta^2 w^2(1 - x^2) + M^2(1 - w)^2\right]^{-1}\,.
\end{cases}$$

It is clear at this stage that although F is independent of η^2, the amplitude $G^{(A)}$ is strongly dependent on η^2 because of the singular nature of the denominators in X and Y. This is of course not surprising; in the spirit of the older approaches to the subject one can point to the resonant intermediate state that can occur in this graph. It is possible to draw a time-ordered graph of the type of Fig. 3a with an intermediate state in which, just as in the initial state, only two nucleons are present. Another way of looking at this is afforded by dispersion theory; for this graph, the cut in the kinetic energy variable extends right down to zero, so one anticipates a strong dependence on η^2.

When we turn to the crossed graph (Fig. 3b), the situation is different. The lowest mass in an intermediate state is $2(M+\mu)$ so the kinetic energy cut starts at

$$\eta'^2 = \tfrac{1}{4}\{4(M + \mu)^2 - 4M^2\} = 2M\mu + \mu^2\,.$$

Thus for energies such that $\eta^2 \ll 2M\mu + \mu^2$, we expect that the dependence of $G^{(B)}$ on η^2 may be ignored. We shall see immediately that this is indeed the case, for

(3.8)
$$
\begin{cases}
G^{(B)} = \dfrac{g^4}{(2\pi)^7} \dfrac{E^2}{M^2} \Lambda^{(-)}(p_2)\gamma_4\gamma'_\mu \Lambda^{(+)}(p_1)(3 + 2\,\tau_p\cdot\tau_n)\,J_{\mu\nu}\Lambda^{(+)}(n_2)\gamma_4\gamma'_\nu \Lambda^{(+)}(n_1)\,, \\[2ex]
J_{\mu\nu} = i\displaystyle\int \mathrm{d}^4 Q\, \dfrac{Q_\mu Q_\nu}{abce}\,, \\[2ex]
e = Q^2 + 2Q\cdot N - \Delta^2 - i\varepsilon\,.
\end{cases}
$$

In Appendix A this is reduced to

(3.9)
$$
G^{(B)} = \frac{g^4}{(2\pi)^7}\,(3 + 2\,\tau_n\cdot\tau_p)\,\frac{1}{M^2}\sum_{i=1}^{5} O_i\, g_i^{(B)}\,,
$$

(3.10)
$$
\begin{cases}
g_1^{(B)} = \Delta^2 F - \dfrac{1}{4}\dfrac{1}{(E + M)^2}\{(E + M)^4 - (K^2 - \Delta^2)^2\}(T - E^2 U)\,, \\[2ex]
g_2^{(B)} = F + \dfrac{1}{2}\dfrac{1}{(E + M)^2}\{(E + M)^2 + K^2 - \Delta^2\}(T - E^2 U)\,, \\[2ex]
g_3^{(B)} = -V - \dfrac{1}{(E + M)^2}(T - E^2 U)\,, \\[2ex]
g_4^{(B)} = T\,, \\[2ex]
g_5^{(B)} = -\Delta^2 T\,,
\end{cases}
$$

(3.11)
$$
\begin{cases}
T = -\dfrac{\pi^2}{8}\displaystyle\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\int_{0}^{1}\mathrm{d}w\,w(1-w)\,[\mu^2 w + \Delta^2 w^2(1-x^2) + \\[1ex]
\hspace{6cm} + K^2(1-y^2)(1-w)^2 + M^2(1-w)^2]^{-1}\,, \\[2ex]
U = -\dfrac{\pi^2}{4}\displaystyle\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\int_{0}^{1}\mathrm{d}w\,w(1-w)^3\,[\mu^2 w + \Delta^2 w^2(1-x^2) + \\[1ex]
\hspace{6cm} + K^2(1-y^2)(1-w)^2 + M^2(1-w)^2]^{-2}\,, \\[2ex]
V = -\dfrac{\pi^2}{4}\displaystyle\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\,y^2\int_{0}^{1}\mathrm{d}w\,w(1-w)^3\,[\mu^2 w + \Delta^2 w^2(1-x^2) + \\[1ex]
\hspace{6cm} + K^2(1-y^2)(1-w)^2 + M^2(1-w)^2]^{-2}\,.
\end{cases}
$$

Now, it is clear that throughout the regions of parametric integration the coefficient of K^2 in the denominators is always less than that of M^2, and since $K^2 \ll M^2$ in the adiabatic limit, we are always justified in setting $K^2 = 0$ in the integrals. This makes the integrals functions of Δ^2 alone, independent of η^2, and accordingly the $g_i^{(B)}$ also depend only on Δ^2.

4. – The iterated Yukawa potential.

The term

$$(4.1) \qquad I = \int d^3k \, (k_2 \, , V^{(1)} | k) \, \frac{1}{W - W_1 - i\varepsilon} \, (k \, | V^{(1)} | k_1) \, ,$$

which occurs in (2.15) is the iterated Yukawa potential. $V^{(1)}$ is given by (3.3) and enables us to write

$$(4.2) \qquad I = \frac{g^4}{(2\pi)^6} \, (3 - 2 \, \tau_p \cdot \tau_n) \, \frac{1}{16 \, M^3} \cdot$$

$$\cdot \int d^3k \, \frac{1}{ABC} \left[\sigma_p \cdot (k_2 - k) \sigma_n \cdot (k_2 - k) \sigma_p \cdot (k_1 - k) \sigma_n \cdot (k_1 - k) \right],$$

$$A = k^2 + \eta^2 + \mu^2 - 2k \cdot k_1 \, ,$$

$$B = k^2 + \eta^2 + \mu^2 - 2k \cdot k_2 \, ,$$

$$C = k^2 - \eta^2 - i\varepsilon \, .$$

It is clear that this integral is linearly divergent; it turns out that only the coefficient of O_1 diverges. In Appendix B, I is reduced to

$$(4.3) \qquad I = \frac{g^4}{(2\pi)^7} \, (3 - 2 \, \tau_p \cdot \tau_n) \, \frac{1}{M^2} \sum_{i=1}^{5} O_i \, j_i \, ,$$

$$(4.4) \qquad \begin{cases} j_1 = - \Delta^2 (X_s + K^2 Y_s) + W, \\[4pt] j_2 = -(X_s + K^2 Y_s) \, , \\[4pt] j_3 = Y_s \, . \\[4pt] j_4 = - X_s \, , \\[4pt] j_5 = \Delta^2 X_s \, , \end{cases}$$

$$(4.5) \begin{cases} W = (2\pi)^2 \frac{Q}{8M} - \frac{(2\pi)^3}{64M} \left[\mu + \frac{1}{2} \varDelta \sin^{-1} \left(\frac{\varDelta}{\sqrt{\varDelta^2 + \mu^2}} \right) \right], \\[2ex]
X_s = \frac{-\pi^2}{8M} \int_{-1}^{1} dx \int_{0}^{1} w \, dw \int_{-\infty}^{\infty} dq \, [q^2 + \mu^2 w + \varDelta^2 w^2(1-x^2) - \eta^2(1-w)^2 - i\varepsilon]^{-1}, \\[2ex]
Y_s = \frac{-\pi^2}{4M} \int_{-1}^{1} dx \int_{0}^{1} w(1-w)^2 \, dw \cdot \\[2ex]
\qquad\qquad \cdot \int_{-\infty}^{\infty} dq \, [q^2 + \mu^2 w + \varDelta^2 w^2(1-x^2) - \eta^2(1-w)^2 - i\varepsilon]^{-2}, \end{cases}$$

where Q is the cut-off momentum.

As with $G^{(4)}$, I is clearly energy-dependent. In the next section we will show that the energy dependence is such as exactly to cancel that of $G^{(4)}$ in the adiabatic limit, which leads to a potential $V^{(2)}$ which is a function of $\varDelta^2$ alone.

5. – The cancellation.

In this section we shall show that the potential defined by (2.14) and (2.15) is energy-independent in the adiabatic limit. As has already been pointed out, the one-meson exchange potential $V^{(1)}$ given by (3.3) is independent of η^2. We now turn to the two-meson exchange potential.

At the end of Section 3, in discussing the contribution of the crossed graph (Fig. 3b), we observed that in the adiabatic limit, the integrals arising there, T, U, and V, are independent of η^2. Indeed, in this approximation,

$$(5.1) \begin{cases} T_{\text{adiabatic}} = \frac{-\pi^2}{8} \int_{-1}^{1} dx \int_{-1}^{1} dy \int_{0}^{1} dw \, w(1-w) \, [\mu^2 w + \varDelta^2 w^2(1-x^2) + M^2(1-w)^2]^{-1}, \\[2ex]
U_{\text{adiabatic}} = \frac{-\pi^2}{4} \int_{-1}^{1} dx \int_{-1}^{1} dy \int_{0}^{1} dw \, w(1-w)^3 \, [\mu^2 w + \varDelta^2 w^2(1-x^2) + M^2(1-w)^2]^{-2}, \\[2ex]
V_{\text{adiabatic}} = \frac{-\pi^2}{4} \int_{-1}^{1} dx \int_{-1}^{1} dy \int_{0}^{1} dw \, y^2 w(1-w)^3 \cdot \\[2ex]
\qquad\qquad \cdot [\mu^2 w + \varDelta^2 w^2(1-x^2) + M^2(1-w)^2]^{-2}, \end{cases}$$

yielding

(5.2)
$$
\begin{cases}
T_{\text{adiabatic}} = F, \\[4pt]
U_{\text{adiabatic}} = 3V_{\text{adiabatic}} = \\[10pt]
\qquad = \dfrac{-\pi^2}{2} \displaystyle\int_{-1}^{1} dx \int_{0}^{1} dw\, w(1-w)^3 \left[\mu^2 w + \Delta^2 w^2(1-x^2) + M^2(1-w)^2\right]^{-2}.
\end{cases}
$$

Substituting in (3.10), and noting further that $\Delta^2 \ll M^2$ and $K^2 \ll M^2$, we find that in the adiabatic limit

(5.3)
$$
\begin{cases}
g_1^{(B)} = M^2 F - M^4 U_{\text{adiabatic}}, \\[6pt]
g_2^{(B)} = \tfrac{3}{2}F - \tfrac{1}{2}M^2 U_{\text{adiabatic}}, \\[6pt]
g_3^{(B)} = -\dfrac{1}{4M^2}\, F + \dfrac{1}{12}\, U_{\text{adiabatic}}, \\[6pt]
g_4^{(B)} = F, \\[6pt]
g_5^{(B)} = -\Delta^2 F.
\end{cases}
$$

Referring to (2.15), let us write

(5.4)
$$
\begin{cases}
V^{(2)} = V^{(A)} + V^{(B)}, \\[4pt]
V^{(A)} = G^{(A)} + I, \\[4pt]
V^{(B)} = G^{(B)}.
\end{cases}
$$

Then $V^{(B)}$ is already of the desired form; i.e., the expressions $V_i^{(B)}$ are functions only of Δ^2. It remains to show that the same holds true for the $V_i^{(A)}$.

(5.5) $V_i^{(A)} = \dfrac{g^4}{(2\pi)^7}\, (3 - 2\,\boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n)\, \dfrac{1}{M^2}\, v_i^{(A)},$

(5.6) $v_i^{(A)} = g_i^{(A)} + j_i,$

(5.7)
$$
\begin{cases}
v_1^{(A)} = \Delta^2(X_c + K^2 Y_c) + W + \dfrac{1}{4(E+M)^2}\{(E+M)^4 - (K^2 - \Delta^2)^2\}F, \\[10pt]
v_2^{(A)} = X_c + K^2 Y_c + \dfrac{1}{2(E+M)^2}\{(E+M)^2 + (K^2 - \Delta^2)\}F, \\[10pt]
v_3^{(A)} = -Y_c - \dfrac{1}{(E+M)^2}\, F, \\[10pt]
v_4^{(A)} = X_c, \\[8pt]
v_5^{(A)} = -\Delta^2 X_c,
\end{cases}
$$

(5.8) $X_c = X - X_s, \qquad Y_c = Y - Y_s.$

From (3.7), in the adiabatic limit when E can be replaced by M,

$$
(5.9) \begin{cases}
X \cong -\dfrac{\pi^2}{8M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}w \; w \int\limits_{-(1-w)M}^{(1-w)M} \mathrm{d}z \left[\mu^2 w + \varDelta^2 w^2(1-x^2) - \eta^2(1-w)^2 + z^2 \right]^{-1}, \\[4mm]
Y \cong -\dfrac{\pi^2}{4M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}w \; w(1-w)^2 \cdot \\[3mm]
\qquad\qquad \cdot \int\limits_{-(1-w)M}^{(1-w)M} \mathrm{d}z \left[\mu^2 w + \varDelta^2 w^2(1-x^2) - \eta^2(1-w)^2 + z^2 \right]^{-2}.
\end{cases}
$$

If it were legitimate to let $M \to \infty$, the z-integration would extend from $-\infty$ to $+\infty$ and X and Y would become X_s and Y_s, the static limits of the respective integrals. However,

$$
(5.10) \begin{cases}
X_c = \dfrac{\pi^2}{4M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}w \; w \int\limits_{M(1-w)}^{\infty} \mathrm{d}z \left[\mu^2 w + \varDelta^2 w^2(1-x^2) - \eta^2(1-w)^2 + z^2 \right]^{-1}, \\[4mm]
Y_c = \dfrac{\pi^2}{2M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}w \; w(1-w)^2 \cdot \\[3mm]
\qquad\qquad \cdot \int\limits_{M(1-w)}^{\infty} \mathrm{d}z \left[\mu^2 w + \varDelta^2 w^2(1-x^2) - \eta^2(1-w)^2 + z^2 \right]^{-2},
\end{cases}
$$

and still in the adiabatic limit, since in these integrals $z \geqslant M(1-w)$, it is legitimate to ignore $\eta^2(1-w)^2$ compared with z^2. Thus we arrive at

$$
(5.11) \begin{cases}
X_c = \dfrac{\pi^2}{4M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}w \; w \int\limits_{M(1-w)}^{\infty} \mathrm{d}z \left[\mu^2 w + \varDelta^2 w^2(1-x^2) + z^2 \right]^{-1}, \\[4mm]
Y_c = \dfrac{\pi^2}{2M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}w \; w(1-w)^2 \int\limits_{M(1-w)}^{\infty} \mathrm{d}z \left[\mu^2 w + \varDelta^2 w^2(1-x^2) + z^2 \right]^{-2},
\end{cases}
$$

and we have thus succeeded in eliminating the η^2 dependence of the integrals. Furthermore,

$$
(5.12) \quad K^2 Y_c < \dfrac{\pi^2}{2M} \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{1}^{1} \mathrm{d}w \; w(1-w)^2 \int\limits_{M(1-w)}^{\infty} \mathrm{d}z \; \dfrac{K^2}{M^2(1-w)^2} \cdot
$$
$$
\cdot \left[\mu^2 w + \varDelta^2 w^2(1-x^2) + z^2 \right]^{-1} = \dfrac{K^2}{M^2} X_c \ll X_c,
$$

The adiabatic approximation involves replacing $1+0(\eta^2/M^2)$ times an expression by the expression. We may accordingly replace $X_c + K^2 Y_c$ by X_c in our expressions. This leads finally to

(5.13)
$$
\begin{cases}
v_1^{(A)} = \Delta^2 X_c + W + M^2 F\,, \\[4pt]
v_2^{(A)} = X_c + \tfrac{1}{2} F\,, \\[4pt]
v_3^{(A)} = - Y_c - \dfrac{1}{4M^2}\, F\,, \\[4pt]
v_4^{(A)} = X_c\,, \\[4pt]
v_5^{(A)} = - \Delta^2 X_c\,,
\end{cases}
$$

where X_c and Y_c as given by (5.11) are independent of η^2, as are W and F.

We would like to stress at this point that our result depends upon the choice of invariants (2.9). For example

$$ O_2 = K^2 - \Delta^2 - i(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n)\cdot\boldsymbol{n} $$

and this depends on η^2 through $K^2 = \eta^2 - \Delta^2$ and through $|\boldsymbol{n}| - K.1$. With a different choice of invariants, this energy dependence might have been thrown on the corresponding V_i, where we would not want it. This point will be further discussed in Section **7**.

6. – Spectral functions.

As is well known, the contributions of processes involving the exchange of n mesons behave like $\exp[-n\mu r]$ in configuration space. Thus, outside a region of radius $1/3\mu$, the dominant contribution to the interaction derives from the exchange of one or of two mesons. This suggests trying to write the potential as a superposition of Yukawa functions of different ranges.

Thus if, in momentum space, we are able to write

(6.1)
$$ V_i(\Delta^2) = \frac{1}{(2\pi)^3}\int d\sigma^2\, \frac{\varrho_i(\sigma^2)}{\sigma^2 + 4\Delta^2}\,, $$

or similar subtracted spectral representations, then, apart from the rescattering corrections to be discussed in the next section, the *only* contributions to the spectral functions ϱ_i in the interval $2\mu < \sigma < 3\mu$ come from $r_i^{(A)}$ and $r_i^{(B)}$.

In this section we throw the potential $r_i^{(A)}$ and $r_i^{(B)}$ into the form (6.1) and give expressions for the corresponding spectral functions. These expressions have been evaluated numerically.

We first derive spectral representations for the integrals F, X_c, Y_c and $U_{\text{adiabatic}}$, e.g.

(6.2)
$$F(\Lambda^2) = \frac{1}{(2\pi)^3} \int\limits_{4\mu^2}^{\infty} d\sigma^2 \, \frac{f(\sigma^2)}{\sigma^2 + 4\Lambda^2} \, ,$$

and similarly for the spectral functions $x(\sigma^2)$, $y(\sigma^2)$ and $u(\sigma^2)$ of X_c, Y_c, and $U_{\text{adiabatic}}$ respectively.

$$F(\Lambda^2) = -\frac{\pi^2}{4} \int\limits_0^1 w(1-w)\,dw \int\limits_{-1}^1 dx \, [\mu^2 w + (1-x^2)w^2\Lambda^2 + M^2(1-w)^2]^{-1} \, ,$$

substituting

(6.3)
$$\sigma^2 = 4\,\frac{\mu^2 w + M^2(1-w)^2}{w^2(1-x^2)} \, ,$$

(6.4) $F(\Lambda^2) = -\dfrac{\pi^2}{4} \displaystyle\int\limits_0^1 w(1-w)\,dw \quad 2\int\limits_{(4/w^2)[\mu^2 w + M^2(1-w)^2]}^{\infty} d\sigma^2 \dfrac{1}{\sigma^2 + 4\Lambda^2} \dfrac{2}{\sigma^2 w^2} \cdot$

$$\cdot \left[1 - \frac{4(\mu^2 w + M^2(1-w)^2)}{w^2\sigma^2}\right]^{-\frac{1}{2}} \, .$$

Interchanging the orders of integration and then performing the w-integration, leads immediately to an expression of the form (6.2) with

(6.5)
$$f(\sigma^2) = -\frac{1}{\mu^2} \frac{(2\pi)^5}{4\tau p^2} \left\{q - \frac{q^2+2}{|p|}\,\Phi\right\} \, ,$$

(6.6)
$$\begin{cases} \tau = \sigma/\mu \, , \\[4pt] q^2 = \tau^2 - 1 \, , \\[4pt] p^2 = 4M^2 - \tau^2 \, , \\[4pt] m = M/\mu \, , \end{cases}$$

(6.7)
$$\begin{cases} \Phi = \cos^{-1}\left\{\dfrac{\frac{1}{2}(q^2+2)}{\sqrt{1+m^2q^2}}\right\} \, , & p^2 > 0 \, , \\[14pt] = \ln\left\{\dfrac{\frac{1}{2}(q^2+2) + \frac{1}{2}q\sqrt{-p^2}}{\sqrt{1+m^2q^2}}\right\} \, , & p^2 < 0 \, . \end{cases}$$

An exactly similar procedure leads to

(6.8)
$$x(\sigma^2) = \frac{1}{\mu^2} \frac{(2\pi)^5}{4\tau^3 m} \left\{ (q^2 + 2) \left(\frac{\pi}{4} - \frac{m}{|p|} \Phi \right) - \Theta \right\},$$

(6.9) (*)
$$\Theta = \operatorname{ctg}^{-1}(mq),$$

(6.10)
$$y(\sigma^2) = \frac{1}{\mu^4} \frac{(2\pi)^5}{\tau^3 m} \left\{ \frac{\tau^2 \Theta}{4} + \frac{mq}{p^2} + \frac{(q^2 + 2)}{\tau^2} \frac{(2p^2 - \tau^2)}{p^2} \frac{m}{|p|} \Phi - \right.$$
$$\left. - \frac{(q^2 + 2)}{\tau^2} \left(\frac{\pi}{2} + \Theta \right) \right\},$$

(6.11)
$$u(\sigma^2) = \frac{1}{\mu^4} \frac{(2\pi)^5}{4\tau_p^4} \left\{ \frac{12(q^2 + 2)}{|p|} \Phi - q \left(8 + \frac{(q^2 + 2)^2}{1 + m^2 q^2} \right) \right\}.$$

Also,

(6.12)
$$W(\varDelta^2) = W(0) - \frac{(2\pi)^3}{128 M} \varDelta^2 \int_{4\mu^2}^{\infty} d\sigma^2 \frac{1}{\sigma} \frac{1}{\sigma^2 + 4\varDelta^2}.$$

We have $W(\varDelta^2)$ in a subtracted form so that the divergence is taken care of; $W(0)$ is of course an (infinite) constant. For this reason it is more convenient to write $v_1^{(A)}$ in a subtracted form and, although it is not strictly necessary, for ease of comparison we do the same for $v_1^{(B)}$. Thus

(6.13)
$$V^{(2)} = \frac{g^4}{(2\pi)^7} \frac{1}{M^2} \sum_{i=1}^{5} O_i \{ (3 - 2 \, \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n) \, v_i^{(A)} + (3 + 2 \, \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n) \, v_i^{(B)} \},$$

(6.14)
$$v_1^{(A,B)} = \operatorname{constant}^{(A,B)} + \frac{(2\pi)^2}{\mu^2} \varDelta^2 \int_{4\mu^2}^{\infty} d\sigma^2 \frac{1}{\sigma^2 + 4\varDelta^2} \mathscr{I}_1^{(A,B)}(\sigma^2),$$

$$v_{2,4}^{(A,B)} = \frac{(2\pi)^2}{\mu^2} \int_{4\mu^2}^{\infty} d\sigma^2 \frac{1}{\sigma^2 + 4\varDelta^2} \mathscr{I}_{2,4}^{(A,B)}(\sigma^2),$$

$$v_3^{(A,B)} = \frac{(2\pi)^2}{\mu^4} \int_{4\mu^2}^{\infty} d\sigma^2 \frac{1}{\sigma^2 + 4\varDelta^2} \mathscr{I}_3^{(A,B)}(\sigma^2),$$

$$v_5^{(A,B)} = \frac{(2\pi)^2}{\mu^2} \varDelta^2 \int_{4\mu^2}^{\infty} d\sigma^2 \frac{1}{\sigma^2 + 4\varDelta^2} \mathscr{I}_5^{(A,B)}(\sigma^2),$$

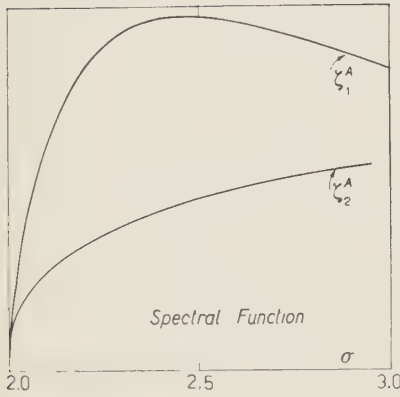(*) This is the same arc cotangent which appeared in the spectral functions derived in I.

Fig. 4. – Spectral functions.

$$(6.15)\qquad
\begin{aligned}
\mathscr{I}_1^{(A)} &= \frac{\mu^2}{(2\pi)^5}\left\{x - \frac{4m^2}{\tau^2}f - \frac{(2\pi)^6}{128\,m\tau}\right\}, \\[2mm]
\mathscr{I}_1^{(B)} &= -\frac{\mu^2}{(2\pi)^5}\frac{4m^2}{\tau^2}\{m^2\mu^2 u - f\}, \\[2mm]
\mathscr{I}_2^{(A)} &= \frac{\mu^2}{(2\pi)^5}\{x + \tfrac{1}{2}f\}, \\[2mm]
\mathscr{I}_2^{(B)} &= \frac{\mu^2}{(2\pi)^5}\tfrac{1}{2}\{m^2\mu^2 u - 3f\}, \\[2mm]
\mathscr{I}_3^{(A)} &= -\frac{\mu^2}{(2\pi)^5}\left\{\frac{f}{4m^2} - \mu^2 y\right\}, \\[2mm]
\mathscr{I}_3^{(B)} &= \frac{\mu^2}{(2\pi)^5}\frac{1}{12}\left\{-\mu^2 u - \frac{3}{4}\frac{f}{m^2}\right\}, \\[2mm]
\mathscr{I}_4^{(A)} &= -\mathscr{I}_5^{(A)}\,\frac{\mu^2}{(2\pi)^5}\,x, \\[2mm]
\mathscr{I}_4^{(B)} &= -\mathscr{I}_5^{(B)} = -\frac{\mu^2}{(2\pi)^5}f.
\end{aligned}$$

These spectral functions have all been calculated numerically, and in Fig. 4 we plot graphs of two of them; $viz.$, the subtracted $\mathscr{I}_1^{(A)}$ and the unsubtracted $\mathscr{I}_2^{(A)}$. In the region of interest, $2\mu < \sigma < 3\mu$, $\mathscr{I}_1^{(B)}$ is very similar in shape to $\mathscr{I}_1^{(A)}$ and all the other spectral functions, despite their different functional forms and despite differences in magnitude, are remarkably similar in shape to $\mathscr{I}_2^{(A)}$. It seems to us superfluous to present so many identical graphs so we give instead a table of values (Table I). In the next section we shall discuss these results.

TABLE I. – *Values of spectral functions.*

σ/μ	2	2.2	2.5	3.0	4.0
$\mathscr{I}_1^{(A)}\cdot 10$	-0.036	1.5	1.7	1.5	0.89
$\mathscr{I}_1^{(B)}\cdot 10^2$	0	1.3	1.5	-1.2	-0.75
$\mathscr{I}_2^{(A)}\cdot 10^4$	0	2.8	4.0	4.9	5.6
$\mathscr{I}_2^{(B)}\cdot 10^4$	0	5.6	8.4	10.2	11.0
$\mathscr{I}_3^{(A)}\cdot 10^6$	0	3.3	-5.0	6.1	6.4
$\mathscr{I}_3^{(B)}\cdot 10^2$	0	4.2	.62	7.6	8.2
$\mathscr{I}_4^{(A)}\cdot 10^4$	0	4.8	7.0	8.5	9.5
$\mathscr{I}_4^{(B)}\cdot 10^4$	0	4.0	6.0	7.2	7.8
$\mathscr{I}_5^{(A)}\cdot 10^4$	0	-4.8	-7.0	8.5	9.5
$\mathscr{I}_5^{(B)}\cdot 10^4$	0	4.0	-6.0	7.2	-7.8

7. – Discussion.

We have shown that for a certain choice of invariant spin functions (2.9), the energy dependence of the two-meson exchange amplitudes arising from the perturbation theory graphs of Figs. 3a, 3b is, for non-relativistic energies, just cancelled by the energy dependence of the iteration of the one-meson exchange amplitude arising from the graph of Fig. 2. This, through the Lippman-Schwinger equation, enables us to define a potential which, apart from the trivial energy dependence entering via the spin functions O_i, is a function only of the momentum transfer; $i.e.$ is local in configuration space. This potential is defined in such a way that on substitution into the non-relativistic Schrödinger equation it will, in perturbation theory, generate a scattering amplitude equal to that arising from the field-theoretic graphs of Figs. 2, 3a, 3b in the adiabatic limit.

At energies below the threshold for meson production, the GNO dispersion relations for the amplitudes $G_i(\eta^2, \Delta^2)$ and unitarity specify the scattering matrix in terms of the zero-energy scattering matrix $G_i(0, \Delta^2)$ (*). Were we to have analogues of the KHURI (5) fixed momentum transfer dispersion relations for the potential theory amplitudes $T_i(\eta^2, \Delta^2)$, these, together with unitarity, would define the potential theory scattering amplitude in terms of $T_i(0, \Delta^2)$. We would then be able to $define$ $T_i(0, \Delta^2) = G_i(0, \Delta^2)$ and, as in I, the formal identity at low energies between the dispersion relations and unitarity for field theory on the one hand, and potential theory on the other, would enable us to conclude that the potential theory scattering amplitude matched that of field theory at non-zero energies. From this, we could in turn conclude that a potential satisfying our conditions exists.

In order to explicitly exhibit this potential, we would have to solve the coupled integral equations, $i.e.$ the dispersion relations and unitarity condition, which define $T_i(\eta^2, \Delta^2)$ in terms of $G_i(0, \Delta^2)$. One might attempt to do this by conventional perturbation theory. Alternatively, if one believed in double dispersion relations for the $T_i(\eta^2, \Delta^2)$ and in a dispersion relation in the momentum transfer variable for $G_i(0, \Delta^2)$, one could use the expansion of the spectral functions as presented in I. In either case, once T_i is known, the determination of the potential V_i is then straight-forward; one uses the fixed momentum transfer dispersion relations for potential scattering. In I, we showed that this formal definition leads to precisely the same potential (for one- and two-meson exchange) as does the naive application of perturbation theory such as we have used in this paper. Our success in demonstrating, in

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(*) It is essential that we use our model with distinguishable nucleons so that the pion contributions to the left-hand cut do not occur.

perturbation theory, the energy cancellation which enables us to define a local potential for the scattering of nucleons with spin, paralleling that for scalar nucleons, makes it extremely plausible that the formal definition of the potential outlined above can also be carried through.

Making the usual guesses about the singularities in the amplitudes of Feynman graphs [8], the Δ^2 cut for a graph involving the exchange of N mesons starts at $\sigma^2 = (N\mu)^2$. Thus, in the interval $\sigma^2 < 9\mu^2$, the only contribution to the spectral functions comes from graphs involving the exchange of one or of two mesons. Apart from the graphs of Figs. 2, 3a, 3b, the only contributions which arise are those from the so-called rescattering corrections to these graphs in which the exchanged meson is multiply scattered by the nucleon before it is reexchanged, those due to radiative corrections to the vertices, and, perhaps most important of all, those from graphs in which there is pion-pion interaction between the exchanged mesons.
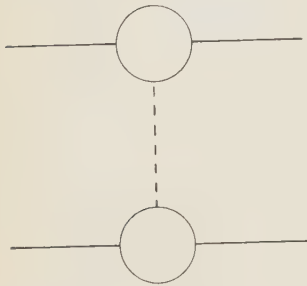

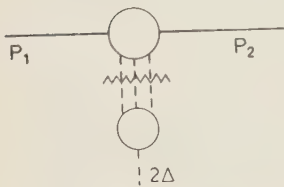Fig. 5. – Corrections to the one pion exchange.

Let us consider these corrections in turn. First there are the corrections to the one-pion exchange graph (Fig. 2) which we symbolize in Fig. 5. These correspond to replacing the vertex $\gamma_5 \tau_i$ by the vertex function $\Gamma_{5i}(p_2, p_1)$. Since, in this diagram, the nucleons are on the mass shell, Γ_{5i} is a function only of $\Delta^2 = \frac{1}{4}(p_2 - p_1)^2$ and so, of course, is the propagator of the exchanged meson. Thus the contribution from the graphs included in Fig. 5 will be a function only of Δ^2. Any of these graphs will, of course, have a pole at $\Delta^2 = -\mu^2/4$ and the effect of summing these is just to renormalize

Fig. 6. – Lowest-mass state contribution to the vertex function.

the coupling constant. However, there is also a cut for $\Delta^2 < -9\mu^2/4$ which comes from the corresponding cut in the vertex function itself, as may be seen from Fig. 6. The lowest mass state in the Δ-variable, illustrated in this diagram, contains three mesons (G-conjugation invariance forbids two mesons). Thus, in the interval $\eta^2 < 9\mu^2$, the only effect is to renormalize the Yukawa potential at $\sigma^2 = \mu^2$.

The vertex corrections to the two-meson exchange uncrossed graph (Fig. 3a)

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[8] G. F. CHEW: Ann. Rev. Nucl. Sci., 9, 29 (1959) surveys the field of the analytic properties of scattering amplitudes in mesodynamics, giving many references to the relevant literature.

are shown in Fig. 7. These will renormalize the coupling constant and, in addition, introduce new spectral functions for $\sigma^2 > 16\mu^2$; this is clear from Fig. 8. In addition, the energy dependence of this graph is altered because of a new cut for $\eta^2 > M\mu + \mu^2/4$ as seen from Fig. 9. However, since we have agreed to the restriction $\eta^2 \ll M\mu + \mu^2/4$, this new energy dependence may be ignored. Exactly similar arguments account for the vertex corrections to Fig. 3b.
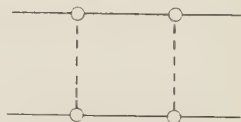
The simplest rescattering graph is that of Fig. 10. The energy dependence of this graph can be ignored for $\eta^2 \ll M\mu + \mu^2/4$ because the threshold for the η^2 cut is again at this point. Other rescattering graphs have cuts starting even higher. Thus the rescattering corrections to the graphs of Fig. 3 are all essentially energy-

Fig. 7. – Vertex corrections to the two-meson exchange uncrossed graph.

independent and the corresponding amplitudes may be added directly to the potential, modifying the spectral function in the region $\sigma^2 > 4\mu^2$ but leaving the local nature of the potential unaffected.
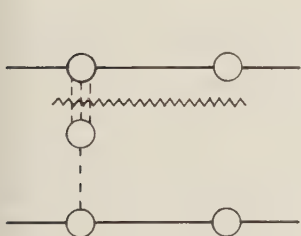

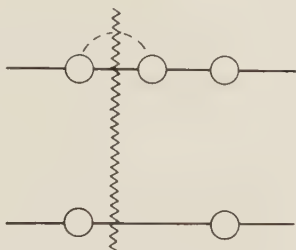Fig. 8. – Lowest mass contribution to new Δ-cut in Fig. 7.

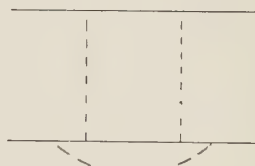Fig. 9. – Lowest-mass contribution to new η-cut in Fig. 7.

Fig. 10. – Simplest rescattering graph.

Finally we have the pion-pion interaction graphs, illustrated in Fig. 11. This has a negligible energy dependence for the same reason as does the crossed graph of Fig. 3b, and again does us no harm.
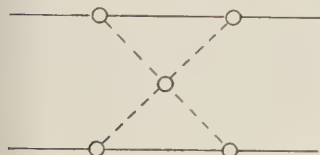

Fig. 11. – Graph involving the pion-pion interaction.

These considerations enable us to conclude that we may still define an energy-independent potential. For, if for the one and two-meson exchange amplitudes we write

(7.1) $$G = G^{(1)} + G^{(A)} + G^{(B)} + G^{(0)} ,$$

then, as we have just seen, $G^{(0)}$ is energy-independent for $\eta^2 \ll M\mu + \mu^2/4$ and using the Lippman-Schwinger equation, the second-order part of the potential

needed to reproduce this is

$$(7.2) \qquad V^{(2)} = G^{(2)} + I \,,$$
$$= G^{(A)} + G^{(B)} + G^{(C)} + I \,.$$

The only energy-dependent terms of $V^{(2)}$ are $G^{(A)}$ and I, whose sum we have already shown to be energy-independent.

We wish at this point to emphasize again that at no stage have we invoked the *static* limit ($\mu/M \to 0$). Indeed, it is clear that this approximation is utterly misleading. The part of the potential which we have called $V^{(A)}$ is a difference between a set of amplitudes calculated from field theory and corresponding amplitudes arising from the iteration of the Yukawa potential. Significant contributions to these amplitudes are the terms involving X and X_s respectively, and the crucial cancellation insures that $X - X_s$ is effectively energy-independent. However, X_s is just the static approximation to X, and had we taken the static limit in the field-theoretic amplitude, we would have had *no* contributions from these terms.

But this is not all. It might be argued that, in any case, the contributions involving X_c are small and those involving Y_c are even smaller. However, the static limit cannot even be applied to the dominant term F (*). This is clear on examination of (6.5)–(6.7). The static limit is obtained by allowing $M/\mu \to \infty$. The spectral function $f(\sigma^2)$ of $F(\Delta^2)$ involves Φ, and in Φ, M/μ occurs in the combination $(M/\mu^2)\sqrt{\sigma^2 - 4\mu^2} - mq$. Thus at the low-mass end of the spectrum ($q \to 0$) the limit $M/\mu \to \infty$ is always ambiguous, just at the point where the static limit would appear to be most justified.

We have, in this paper, paid scant attention to the problem of exchange forces. Since these are discussed in detail in II, we shall do no more here than summarize the method developed there for their treatment. We have calculated the potential as though the two particles involved in the scattering were distinguishable, although each was given an isotopic spin. This has led to a potential which is dependent on isotopic spin and which, we contend, will generate a scattering amplitude the same as would be obtained in a field theory for the scattering of two *distinguishable* nucleons interacting through the meson field. If this scattering amplitude be now antisymmetrized, we claim it will be just that for the scattering of indistinguishable nucleons (if we ignore pair effects which can only contribute to the extremely short-range part of the potential).

(*) This fact has been derived independently by S. N. GUPTA: *Phys. Rev.*, **117**, 1146 (1960) in which he derives a two-meson exchange potential for nuclear scattering. Our potential differs from his because he drops the $(p - q)^2$ term in the denominators of his equation (15). *We do not think that this is permissible.*

The results of this paper are not as satisfactory or complete as were those of I. We have here had to rely far too heavily on perturbation theory. As we have remarked earlier, we are unable to present our results independently of perturbation theory since this would require something which we at present lack, *viz.* a statement of the double dispersion relations for the scattering amplitudes in non-relativistic potential theory for particles with spin. The difficulties in deriving such relations are considerable; even the one-dimensional dispersion relation has so far only been generalized to include tensor forces [9]. The inclusion of the spin-orbit interaction would seem to constitute the major stumbling block.

We do not doubt that double dispersion relations for such potential scattering are valid, but they probably are true only for the amplitudes corresponding to an appropriately restricted choice of basic spin invariants. The set which we have used, (2.9), may well be an appropriate choice; in any case, we can already rule out many possible sets which do not work even to fourth order, *e.g.* that obtained by using unit vectors in (2.9). Once given the double dispersion relations, the parallel between the scattering of spinless nucleons and that of nucleons with spin could be continued and, as in I, a rigorous constructive definition of the two-nucleon potential could be derived.

At present we have no way of calculating the contributions from the rescattering graphs or from those involving pion-pion interactions. Semi-phenomenological estimates utilizing resonances are possible, although perhaps premature. We have, however, given a prescription for the inclusion of such effects. When this is done, we claim that the potential derived should be adequate for the description of nucleon-nucleon scattering for energies below that of the meson production threshold.

Appendix A

Calculation of field-theoretic two-meson exchange graphs.

We first show how (3.5) and (3.6) follow from (3.4):

$$(A.1) \qquad I_{\mu\nu} = i\frac{3}{2}\int \mathrm{d}^4Q \int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y \int_{0}^{1}\mathrm{d}w\, w(1-w)[Q^2 - 2S\cdot Q + \Lambda]^{-4}Q_\mu Q_\nu,$$

[9] J. Hamilton: *Phys. Rev.*, **114**, 1170 (1959).

where

$$
\begin{cases}
S = \Delta x w - (K + Cy)(1 - w)\,, \\
\quad = \mu^2 w + \Delta^2 (2w - 1) - i\varepsilon\,, \\
\Delta = \tfrac{1}{2}(p_2 - p_1)\,, \\
K = \tfrac{1}{2}(P - N)\,, \\
C = \tfrac{1}{2}(P + N)\,,
\end{cases}
$$

(A.2)

$$
\begin{cases}
I_{\mu\nu} = i\,\dfrac{3}{2}\int \mathrm{d}^c\tilde{Q} \int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y \int_{0}^{1}\mathrm{d}w\, w(1-w)\,[\tilde{Q}^2 + \Delta - S^2]^{-4}(\tilde{Q}_\mu \tilde{Q}_\nu + S_\mu S_\nu)\,, \\[2mm]
\quad = -\dfrac{3}{2}\dfrac{\pi^2}{12}\int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y \int_{0}^{1}\mathrm{d}w\, w(1-w)\{(\Delta - S^2)^{-1}\delta_{\mu\nu} + 2(\Delta - S^2)^{-2}S_\mu S_\nu\}\,,
\end{cases}
$$

(A.3)

(A.4) $\quad \Lambda^{(+)}(p_2)\gamma_4\gamma_\mu \Lambda^{(+)}(p_1)\Lambda^{(+)}(n_2)\gamma_4\gamma_\nu \Lambda^{(+)}(n_1)\dfrac{E^2}{M^2}\,\delta_{\mu\nu} =$

$$
= \left\{ \Delta^2 + \frac{1}{4(E+M)^2}\,[(E+M)^4 - (K^2 - \Delta^2)^2] \right\}\frac{1}{M^2}\,O_1 +
$$

$$
- \left\{ 1 + \frac{1}{2(E+M)^2}\,[(E+M)^2 + (K - \Delta^2)] \right\}\frac{1}{M^2}\,O_2
$$

$$
- \frac{1}{(E+M)^2}\,\frac{1}{M^2}\,O_3 + \frac{1}{M^2}\,O_4 - \frac{\Delta^2}{M^2}\,O_5\,,
$$

(A.5) $\quad \Lambda^{(+)}(p_2)\gamma_4\gamma_\mu \Lambda^{(+)}(p_1)\Lambda^{(+)}(n_2)\gamma_4\gamma_\nu \Lambda^{(+)}(n_1)\dfrac{E^2}{M^2}\,S_\mu S_\nu =$

$$
= \left\{ K^2\Delta^2 - \frac{E^2 y^2}{4(E+M)^2}\,[(E+M)^4 - (K^2 - \Delta^2)^2] \right\}(1-w)^2\,\frac{1}{M^2}\,O_1 +
$$

$$
+ \left\{ K^2 - \frac{E^2 y^2}{2(E+M)^2}\,[(E+M)^2 + K^2 - \Delta^2] \right\}(1-w)^2\,\frac{1}{M^2}\,O_2 +
$$

$$
+ \left\{ -1 + \frac{E^2 y^2}{(E+M)^2} \right\}(1-w)^2\,\frac{1}{M^2}\,O_3 + \text{terms odd in } y\,.
$$

Define

$$
\begin{cases}
X = -\dfrac{\pi^2}{8}\int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y \int_{0}^{1}\mathrm{d}w\, w(1-w)(\Delta - S^2)^{-1}\,, \\[3mm]
Y = -\dfrac{\pi^2}{4}\int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y \int_{0}^{1}\mathrm{d}w\, w(1-w)^3(\Delta - S^2)^{-2}\,, \\[3mm]
Z = -\dfrac{\pi^2}{4}\int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y\, y^2 \int_{0}^{1}\mathrm{d}w\, w(1-w)^3(\Delta - S^2)^{-2}\,,
\end{cases}
$$

(A.6)

where $\Delta - S^2 = \mu^2 w + \Delta^2 w^2(1 - x^2) - \eta^2(1-w)^2 + E^2 y^2(1-w)^2 - i\varepsilon$.

Then X and Y are just as defined in (3.7). Furthermore, integrating by parts on y in X yields the relationship

(A.7) $$X = E^2 Z + F$$

and then (3.5) and (3.6) follow immediately.

Similarly, from (3.8),

(A.8) $$J_{\mu\nu} - i\frac{3}{2}\int \mathrm{d}^4 Q \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y \int_{0}^{1} \mathrm{d}w\, w(1-w)[Q^2 - 2R\cdot Q + \Lambda]^{-4} Q_\mu Q_\nu,$$

where $R = \Lambda x w + (Ky + C)(1-w)$,

(A.9) $$J_{\mu\nu} - -\frac{3}{2}\frac{\pi^2}{12}\int_{-1}^{1}\mathrm{d}x \int_{-1}^{1}\mathrm{d}y \int_{0}^{1}\mathrm{d}w\, w(1-w)\{(\Lambda - R^2)^{-1}\delta_{\mu\nu} + 2(\Lambda - R^2)^{-2} R_\mu R_\nu\},$$

(A.10) $$\Lambda^{(+)}(p_2)\gamma_4\gamma_\mu\Lambda^{(+)}(p_1)\Lambda^{(+)}(n_2)\gamma_4\gamma_\nu\Lambda^{(+)}(n_1)\frac{E^2}{M^2}R_\mu R_\nu =$$

$$= \left\{K^2\Lambda^2 y^2 - \frac{E^2}{4(E+M)^2}[(E+M)^4 - (K^2-\Lambda^2)^2]\right\}(1-w)^2\frac{1}{M^2}O_1 +$$

$$+ \left\{K^2 y^2 - \frac{E^2}{2(E+M)^2}[(E+M)^2 + K^2 - \Lambda^2]\right\}(1-w)^2\frac{1}{M^2}O_2 +$$

$$+ \left\{-y^2 + \frac{E^2}{(E+M)^2}\right\}(1-w)^2\frac{1}{M^2}O_3 + \text{terms odd in } y,$$

with

(A.11) $$\Lambda - R^2 = \mu^2 w + \Lambda^2 w^2(1-w^2) + K^2(1-y^2)(1-w)^2 + M^2(1-w)^2.$$

Equations (3.9) and (3.10) follow from the above if T, U, and V are defined as in (3.11) and use is made of the identity

(A.12) $$T = F - K^2 V,$$

which follows by partial integration on y in T.

Appendix B

Calcultion of the iterated Yukawa potential.

We start from equation (4.2)

(B.1)
$$\left\{ \begin{array}{l} I - \dfrac{g^4}{(2\pi)^6}\,(3 - 2\,\boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n)\,\dfrac{1}{16M^3}\cdot \\[2mm] \qquad\qquad \cdot \displaystyle\int d^3k\,\dfrac{1}{ABC}\,[\boldsymbol{\sigma}_p \cdot (\boldsymbol{k}_2 - \boldsymbol{k})\,\boldsymbol{\sigma}_n \cdot (\boldsymbol{k}_2 - \boldsymbol{k})\,\boldsymbol{\sigma}_p \cdot (\boldsymbol{k}_1 - \boldsymbol{k})\,\boldsymbol{\sigma}_n \cdot (\boldsymbol{k}_1 - \boldsymbol{k})]\,, \\[3mm] - \dfrac{g^4}{(2\sigma)^7}\,(3 - 2\,\boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_n)\,\dfrac{1}{M^2}\,J\,, \end{array} \right.$$

(B.2)
$$J - \frac{\pi}{8M}\int d^3k \int_{-1}^{1} dx \int_{0}^{1} dw\, w\,[k^2 + \eta^2(2w - 1) + \mu^2 w - 2\boldsymbol{k}\cdot(\boldsymbol{K} - \boldsymbol{\Delta}x)w - i\varepsilon]^{-3}\cdot$$
$$\cdot\{[(\boldsymbol{K} - \boldsymbol{k})^2 - \varDelta^2]^2 - 2i(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n)\cdot(\boldsymbol{n} - \boldsymbol{k}\times\boldsymbol{\Delta})[(\boldsymbol{K} - \boldsymbol{k})^2 - \varDelta^2] -$$
$$- 4\boldsymbol{\sigma}_p\cdot(\boldsymbol{n} - \boldsymbol{k}\times\boldsymbol{\Delta})\boldsymbol{\sigma}_n\cdot(\boldsymbol{n} - \boldsymbol{k}\times\boldsymbol{\Delta})\}\,.$$

Although this integral is linearly divergent, since it is only three-dimensional, no surface terms are introduced by making a translation in the $\boldsymbol{k}$ integration. Thus we may write

(B.3)
$$\boldsymbol{q} = \boldsymbol{k} - (\boldsymbol{K} - \boldsymbol{\Delta}x)w\,,$$

(B.4)
$$J = \frac{\pi}{8M}\int d^3q \int_{-1}^{1} dx \int_{0}^{1} dw\, w\,[q^2 + \mu^2 w + \varDelta^2 w^2(1 - x^2) - \eta^2(1 - w)^2 - i\varepsilon]^{-3}\cdot$$
$$\cdot\{q^4 + 2q^2[K^2(1 - w)^2 - \varDelta^2 + \varDelta^2 x^2 w^2] + 4\,[\boldsymbol{q}\cdot(\boldsymbol{K}(1 - w) - \boldsymbol{\Delta}x\,w)]^2 +$$
$$+ [K^2(1 - w)^2 - \varDelta^2 + \varDelta^2 x^2 w^2]^2 - 2i(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n)\cdot\boldsymbol{n}(1 - w)\,[q^2 + K^2(1 - w)^2 -$$
$$- \varDelta^2 + \varDelta^2 w^2 x^2] - 4i[(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n)\cdot\boldsymbol{q}\times\boldsymbol{\Delta}]\,\boldsymbol{q}\cdot[\boldsymbol{K}(1 - w) - \boldsymbol{\Delta}x\,w] -$$
$$- 4(\boldsymbol{\sigma}_p\cdot\boldsymbol{n})(\boldsymbol{\sigma}_n\cdot\boldsymbol{n})(1 - w)^2 - 4[\boldsymbol{\sigma}_p\cdot\boldsymbol{q}\times\boldsymbol{\Delta}][\boldsymbol{\sigma}_n\cdot\boldsymbol{q}\times\boldsymbol{\Delta}]\, + \text{terms odd in } \boldsymbol{q}\}\,.$$

Since the denominator is independent of the direction of $\boldsymbol{q}$, the terms $q_i q_j$ in the numerator may be replaced by $\frac{1}{3}q^2\delta_{ij}$, leaving as effective numerator to replace $\{...\}$:

(B.5)
$$\{q^4 + \tfrac{10}{3}q^2[K^2(1 - w)^2 + \varDelta^2 x^2 w^2 - (1 - w)(K^2 - \varDelta^2) - \tfrac{3}{5}\varDelta^2] +$$
$$+ [K^2(1 - w)^2 - \varDelta^2 + \varDelta^2 x^2 w^2][K^2(1 - w)^2 - \varDelta^2 + \varDelta^2 x^2 w^2 - 2(1 - w)(K^2 - \varDelta^2)]\}O_1 +$$
$$- \{\tfrac{10}{3}q^2(1 - w) + 2[K^2(1 - w)^2 - \varDelta^2 + \varDelta^2 x^2 w^2](1 - w)\}O_2 -$$
$$- 4(1 - w)^2 O_3 + \tfrac{4}{3}q^2 O_4 - \tfrac{4}{3}q^2\varDelta^2 O_5\,.$$

The angular integration is now trivial and results in

(B.6)
$$J = \sum_{i=1}^{5} O_i j_i \,,$$

(B.7) $$j_i = \frac{\pi^2}{2M} \int_0^\infty dq \, q^2 \int_{-1}^1 dx \int_0^1 dw \, w \, (\text{numerator})_i \cdot$$

$$\cdot [q^2 + \mu^2 w + \Delta^2 w^2 (1 - x^2) - \eta^2 (1 - w)^2 - i\varepsilon]^{-3} \,,$$

where (numerator), is the coefficient of O_i in (B.5). A rather lengthy and involved series of integrations by parts, the details of which will be omitted, leads to the results cited in (4.4).

RIASSUNTO (*)

Si discute lo scattering dei nucleoni ad energie inferiori alla soglia di produzione dei mesoni; l'ampiezza è una somma di cinque termini, ciascuno proporzionale ad un differente invariante di spin. Un confronto nella teoria della perturbazione fra l'ampiezza di campo teoretico e quella derivante dalla teoria del potenziale permette di concludere che esiste un potenziale che riproduce l'ampiezza di campo teoretico tramite le contribuzioni di scambio di due mesoni. Anche questo potenziale è una somma di cinque termini ed è « locale » nel senso che, con una opportuna scelta degli operatori derivanti dallo spin, i suoi coefficienti sono indipendenti dall'energia. Si deduce il potenziale a prescindere da correzioni di radiazione o riscattering; si osserva ancora che il limite statico ($\mu/M \cdot 0$) non è ben definito. Si calcolano le funzioni spettrali per il potenziale. Si discutono le correzioni di riscattering, ecc., e, postulando semplici proprietà di analiticità per i diagrammi di Feynman, si dimostra che non distruggono la localizzazione del potenziale. Non è ancora possibile dare una prova, indipendente dalla teoria della perturbazione, dell'esistenza del potenziale, come è stato fatto per particelle senza spin nella parte I, perchè non sono state ancora provate le necessarie relazioni di dispersione; infatti non è ancora noto per quale scelta degli invarianti di spin ci si può aspettare che esse siano valide. Alcuni indizi su questa scelta di « adatti » invarianti di spin vengono forniti dalle limitazioni imposte perchè il potenziale di quarto ordine sia indipendente dall'energia.

Properties of the Momentum Space Analytic Function.

H. ARAKI (*)

Department of Physics, University of Illinois - Urbana, Ill.

N. BURGOYNE

Department of Mathematics, Princeton University - Princeton, N.J.

Summary. — Using only the fundamental assumptions of Lorentz covariance, locality, and mass spectrum we derive certain results concerning the analytic function in momentum space. The boundary values of this function are closely related to the scattering amplitudes.

In field theory the n-fold analytic function in momentum space is obtained by studying the retarded functions [1]

$$(1) \qquad r_1(x_1, ..., x_n) = \sum \pm \, \theta_\pm(i_1 i_2)\, \theta_\pm(i_2 i_3) ... \theta_\pm(i_{n-1} i_n)\, W(i_1, ..., i_n)\,,$$

where

$$(2) \qquad W(i_1, ..., i_n) = \langle 0 | A_{i_1}(x_{i_1}) ... A_{i_n}(x_{i_n}) | 0 \rangle\,.$$

(*) On leave from the Department of Nuclear Engineering, Kyoto University, Kyoto, Japan.

[1] H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **6**, 319 (1957); V. GLASER, H. LEHMANN and W. ZIMMERMANN: *Nuovo Cimento*, **6**, 1122 (1957); see also G. KÄLLÉN: *Dan. Mat. Fys. Medd.*, **27**, no. 12 (1953). Our notation and that of the first named authors are related as follows: $r_j(x_1, ..., x_n) = (i)^{n-1} r(x_j; x_1, ..., x_n)$. Formula (1) is equivalent to the definition in terms of multiple commutators.

The $A_i(x_i)$ are relativistically covariant [2], local fields, satisfying suitable spectrum conditions [3,4]. In formula (1) $\theta_{\pm}(jk) = \theta(\pm(x_j - x_k))$, and the sign choice for each θ is uniquely determined by the requirement that x_1 be advanced with respect to all the other x_i. The overall sign for each term depends on whether there are an even ($+$) or odd ($-$) number of «$-$» subscripts. The sum is over all permutations of $(1\,2\,...\,n)$. Since r_1 vanishes unless all x_i $(i = 2, ..., n)$ are retarded with respect to x_1 its Fourier transform $\tilde{r}_1(p_1, ..., p_n)$ can be extended to a function $H(s_1, ..., s_n)$, analytic for $q_2, q_3, ..., q_n \in V_-$, where $s = p + iq$ and V is the interior of the past light cone. Due to translation invariance in x-space $\tilde{r}_1$ and H are only defined for $s_1 + s_2 ... + s_n = 0$.

Recently a number of people have independently investigated in detail the properties of H that follow solely from relativistic covariance, locality and mass spectrum. In the 3-fold case the situation is well known [4]. The 4-fold case was first fully explained by STEINMANN [5], who also obtained important results for the general case. RUELLE [6], has, for the n-fold case, derived a simple characterization of the domain of analyticity of H. He also has indicated methods for obtaining many of the general properties of H. Independently of each other and using different approaches the present authors have also investigated the n-fold case. Our results go beyond those of STEINMANN in that we consider all possible boundary values of H (and give their explicit description and properties). The inverse Fourier transform of these boundary values will be generalizations of the usual retarded functions (1). We have found (see also STEINMANN [5]) the necessary and sufficient conditions on these functions for them to be obtainable from the W-functions (2). These conditions are a) relativistic covariance, b) retardedness in certain variables in x-space, c) identities among them in p-space for real momenta below threshold, and d) certain, identically satisfied, linear relations among them (the Steinmann relations). From these generalized retarded functions we also obtain directly the results of Ruelle and extend them to include the details of the mass spectrum. As an example of our results we now give explicit formulas for the physically interesting cases $n = 4, 5$. Our results for the general case will be published later.

For the 4-fold case, the boundary values of $H(s_1, ..., s_4)$ are denoted by $\tilde{r}_j$, $\tilde{a}_j$, $\tilde{r}_{jk}$, $\tilde{a}_{jk}$ as the q's approach zero from inside the cones $V_j(4-)$, $V_j(4+)$, $V_{jk}(4-)$

[2] As written, (1) holds only for commuting fields. If spinor fields are included the factor $\sigma(i_1, ..., i_n) =$ sign of permutation of the anticommuting fields must be included on the right hand side of formula (2).

[3] A. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956); G. KÄLLÉN and A. WIGHTMAN: *Mat. Fys. Skr. Dan. Vid. Selsk.*, no. 6 (1958).

[4] R. JOST: *Helv. Phys. Acta*, **21**, 263 (1958).

[5] O. STEINMANN: *Thesis*, Zurich, 1959 (unpublished); *Helv. Phys. Acta* (to appear).

[6] D. RUELLE: *Thesis*, Bruxelles, 1959 (unpublished).

$V_{jk}(4+)$ respectively. The various cones are defined by

(3)
$$
\begin{cases}
V_j\,(4\,\pm) = \text{all } q \text{ such that } q_k,\ q_m,\ q_n \in V_\pm\,, \\
V_{jk}(4\,\pm) = \text{all } q \text{ such that } -q_k,\ (q_k+q_m),\ (q_k+q_n) \in V_\pm\,,
\end{cases}
$$

where $(jkmn)$ = permutation of (1234). Here $\tilde r_j$ and $\tilde a_j$ are the Fourier transforms of the usual retarded and advanced functions, and $\tilde r_{jk}$ is the Fourier transform of a function $r_{jk}(x_1, ..., x_4)$ defined by the right hand side of (1) where now, however, the subscripts of the θ's are (uniquely) determined by the requirement that x_j be advanced over x_m and x_n, while x_k be advanced over either x_m or x_n. To obtain $\tilde a_{jk}$ read « retarded » instead of « advanced » in the previous sentence.

H will be analytic at all points obtained by applying a complex Lorentz transformation to any point $(s_1, ..., s_4)$ with its imaginary part $(q_1, ..., q_4)$ in one of the cones (3). We note that this domain of analyticity for H can quite probably be further extended by a (non-trivial) analytic continuation as was done for the massless 3-fold case by KÄLLÉN and WIGHTMAN ([3]). The 32 boundary values given by (3) are not linearly independent but are restricted by 6 identically satisfied Steinmann relations

(4) $\tilde r_{jk}(p_1, ..., p_4) + \tilde r_{kj}(p_1, ..., p_4) = \tilde a_{mn}(p_1, ..., p_4) + \tilde a_{nm}(p_1, ..., p_4)\,.$

Letting M_k, $M_{jm}\ (= M_{mj} = M_{kn})$, be the relevant thresholds the mass spectrum implies the following conditions,

(5)
$$
\begin{cases}
\text{if }\ p_k^2 < M_k^2 \qquad\qquad \text{then}\quad \tilde r_j\,(p_1, ..., p_4) = \tilde r_{jk}(p_1, ..., p_4)\,, \\
\text{if }\ (p_j + p_m)^2 < M_{jm}^2 \quad \text{then}\quad \tilde r_{jk}(p_1, ..., p_4) = \tilde a_{mn}(p_1, ..., p_4)\,.
\end{cases}
$$

The metric is $(+\,-\,-\,-)$.

For the 5-fold case, the boundaryv alueso f $H(s_1, ..., s_5)$ divide into six classes, $\tilde r_j$, $\tilde r_{jk}$, $\tilde r'_{jk}$, $\tilde r_{jk,m}$, $\tilde r'_{jk,m}$, and $\tilde r_{j,km}$: They are obtained as the q's approach zero from inside the cones,

(6)
$$
\begin{cases}
V_j(5\,-) \quad\ = \text{all } q \text{ such that } q_k,\ q_m,\ q_n,\ q_t \in V_-\,, \\
V_{jk}(5\,-) \quad = \text{all } q \text{ such that } -q_k,\ (q_k+q_m),\ (q_k+q_n),\ (q_k+q_t) \in V_-\,. \\
V'_{jk}(5\,-) \quad = \text{all } q \text{ such that } (q_j+q_m),\ (q_j+q_n),\ (q_j+q_t),\ (q_k+q_m)\,, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad (q_k+q_n),\ (q_k+q_t),\ \in V_+\,, \\
V_{jk,m}(5\,-) = \text{all } q \text{ such that } q_m,\ -(q_k+q_m),\ (q_k+q_n),\ (q_k+q_t) \in V_-\,, \\
V'_{jk,m}(5\,-) = \text{all } q \text{ such that } -(q_k+q_m),\ (q_j+q_m),\ (q_k+q_n),\ (q_k+q_t) \in V_+\,, \\
V_{j,km}(5\,-) = \text{all } q \text{ such that } q_n,\ q_t,\ (q_j+q_k),\ (q_j+q_m) \in V_-\,,
\end{cases}
$$

where $(jkmnt) = $ permutation of (12345). Similar definitions hold for the corresponding advanced functions and the cones $V(5+)$. There are a total of 370 such functions. As before, r_j are the usual retarded functions, while the other generalized retarded functions are defined in a manner exactly analogous to the 4-fold case. Reading « advanced over » for the symbol « $>$ », their supports are,

(7)
$$\begin{cases} r_{jk}: & x_j > x_m, \, x_n, \, x_t \quad \text{and} \quad x_k > x_m \text{ or } x_n \text{ or } x_t, \\ r_{jk,m}: & x_j > x_n, \, x_t, \quad x_k > x_n \text{ or } x_t \quad \text{and} \quad x_j \text{ or } x_k > x_m, \end{cases}$$

for r'_{jk}, $r'_{jk,m}$ and $r_{j,km}$ similar relations hold (^{7}). However these last three types can also be defined directly in terms of the r_{jk} and $r_{jk,m}$ by using the Steinmann relations,

(8)
$$\begin{cases} \tilde{r}_{jk}(p_1, ..., p_5) + \tilde{r}'_{jk,m}(p_1, ..., p_5) = \tilde{r}_{jk,n}(p_1, ..., p_5) + \tilde{r}_{jk,t}(p_1, ..., p_5), \\ \tilde{r}'_{jk}(p_1, ..., p_5) + \tilde{r}_{jk,m}(p_1, ..., p_5) = \tilde{r}'_{jk,n}(p_1, ..., p_5) + \tilde{r}'_{jk,t}(p_1, ..., p_5), \\ \tilde{r}'_{kj}(p_1, ..., p_5) + \tilde{r}_{m,jk}(p_1, ..., p_5) = \tilde{r}'_{jk,m}(p_1, ..., p_5) + \tilde{r}'_{kj,m}(p_1, ..., p_5). \end{cases}$$

There are similar equations for the advanced functions. Altogether these 220 independent relations reduce the number of linearly independent boundary values to 150. To complete the 5-fold case we give the conditions that follow from the mass spectrum. As before if M_k and M_{km} ($=M_{mk}$) (^{8}) represent the various thresholds then

(9)
$$\begin{cases} \text{if} \quad p_k^2 < M_k^2 & \text{then} \quad \tilde{r}_j = \tilde{r}_{jk} \quad \text{and} \quad \tilde{r}_{jm,k} = \tilde{a}_{j,nt}, \\ \text{if} \quad (p_k + p_m)^2 < M_{km}^2 & \text{then} \quad \tilde{r}_{jk} = \tilde{r}_{jk,m}, \quad \tilde{r}'_{jk} = \tilde{r}'_{jk,m}, \\ & \text{and} \quad \tilde{r}_{jk,n} = \tilde{r}'_{jk,t}, \quad \tilde{r}'_{kj,m} = \tilde{r}_{m,kj}. \end{cases}$$

The time-ordered function can be expressed in terms of the analytic function H. Let

(10)
$$\tau(x_1, ..., x_n) = \sum \theta_+(i_1 i_2) \, \theta_+(i_2 i_3) \, ... \, \theta_+(i_{n-1} i_n) \, W(i_1, ..., i_n)$$

then $\tilde{\tau}(p_1, ..., p_n)$ its Fourier transform is the boundary value of H as $(q_1, ... q_n)$ approaches zero from inside the cone in which $(p_1, ..., p_n)$ lies. If some p_j or

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(^{7}) An alternative way of determining the $\pm$ subscripts for the θ's is to take $\theta_\pm(i_k i_{k+1})$ according whether $(q_{i_1} + ... + q_{i_k}) \, \varepsilon \, V_\pm$ for q in the V's of (3) and (6).

(^{8}) We note that these thresholds will in general differ from those in the 4-fold case, to avoid unnecessarily complicating the notation we however use the same symbols.

$(p_j + p_k)$ etc., are space like or below threshold then by (5) or (9) the corresponding q_j or $(q_j + q_k)$, etc., can approach zero in either V_+ or $\overline{V}_-$. Should one assume the asymptotic condition, then the scattering amplitude is directly connected with the boundary values of H. Finally we mention that the W-function (2) (with all vacuum structure truncated) can be expressed in terms of the generalized retarded functions in a manner similar to equation (1).

* * *

RIASSUNTO (*)

Facendo uso solamente delle ipotesi fondamentali della covarianza di Lorentz, località, e spettro di massa deriviamo alcuni risultati concernenti la funzione analitica nello spazio degli impulsi. I valori ai limiti di questa funzione sono strettamente connessi alle ampiezze di scattering.

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(*) *Traduzione a cura della Redazione.*

Coherence Properties of Blackbody Radiation.

R. C. Bourret

Hughes Research Laboratories, A Division of Hughes Aircraft Company - Malibu, Cal.

Summary. — An explicit calculation of the space and time correlation functions of the electric field components in blackbody radiation is given. These functions are determined by the statistics of a photon gas since the Planck distribution law is essentially the power spectrum of the field fluctuations. The spatial coherence functions are developed in close analogy with the theory of isotropic turbulence of an incompressible fluid.

1. – Introduction.

In recognition of the operationalism of much of contemporary physics it has lately been stressed, notably by Wolf [1], that the true observables of optics are not the fluctuating microfields themselves but various quadratic averages of them which are purveyed to us by detectors. The observables of greatest interest at the present time are the choerence functions of the electric fields, *i.e.*,

$$\langle E_i(r_1, t_1,) \, E_j(r_2, t_2) \rangle \, ,$$

where the brackets signify an average over time. The canonical detectors for the determination of these quantities are polarizers, phase-plates and interferometers. Generally such coherence functions depend upon the properties of their sources (natural linewidth, polarization, line broadening, etc.) and upon the geometrical relations between source and detector. There is one case, however, in which these coherence functions depend solely upon the intrinsic statistical properties of the photons themselves (*i.e.*, Bose-Einstein

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[1] E. Wolf: *Nuovo Cimento*, **12**, 884 (1954).

statistics) and the relative co-ordinates $|\boldsymbol{r}_2 - \boldsymbol{r}_1|$ and $|t_1 - t_2|$ of the coherence functions, namely, the case of blackbody radiation. We believe that this special case is of sufficient interest to warrent the explicit evaluation of the spatial and temporal coherence functions, which are developed in the following pages.

Consider the radiation field enclosed in a cavity with whose perfectly absorbing walls it is in thermal equilibrium. Its energy density (in Gaussian units) is

$$(1) \qquad U = \frac{1}{8\pi} (\; \boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) \; .$$

We assume the cavity to be a vacuum in which (apart from dimensions)

$$(2) \qquad \boldsymbol{D} = \boldsymbol{E} , \qquad \boldsymbol{B} = \boldsymbol{H} ,$$

and

$$(3) \qquad \operatorname{div} \boldsymbol{H} = \operatorname{div} \boldsymbol{E} = 0 \; .$$

These last relations will be required later. We assume, moreover, that the radiation is isotropic and homogeneous, and that

$$(4) \qquad \langle E^2 \rangle = \langle H^2 \rangle \; .$$

The bracket symbol indicates an ensemble average at the same position and time in a large number of equivalent cavities. We shall require the ergodic hypothesis, too, since we shall need to regard the ensemble average as equal to the time and to the space average of fluctuating field quantities. The isotropicity of the electric field may be stated

$$(5) \qquad \langle E_i E_j \rangle = \langle E_i^2 \rangle \, \delta_{ij} \qquad\qquad \text{(no summation).}$$

With (2) and (1) equation (5) becomes

$$(6) \qquad \langle E_i E_j \rangle = \frac{4\pi}{3} \, \delta_{ij} \, U \; .$$

2. – Coherence functions and the wave-equation.

The quantities we wish to derive are the cross-correlation functions of the electric field components, namely,

$$(7) \qquad \langle E_i(\boldsymbol{r}_1, t_1) \, E_j(\boldsymbol{r}_2, t_2) \rangle \; .$$

If now, stationarity in space and in time is assumed, these quantities depend only upon the spatial and temporal differences $r = r_2 - r_1$ and $t = t_2 - t_1$. Making this assumption, we define the quantities of interest by

$$(8) \qquad \mathscr{E}_{ij}(r, t) = \langle E_i(r_1, t_1)\, E_j(r_2, t_2) \rangle \ .$$

WOLF ([1]) has described quantities similar to our $\mathscr{E}_{ij}(r, t)$ which presuppose only temporal stationarity of the field fluctuations. Thus he writes

$$(9) \qquad \mathscr{E}_{ij}(r_1, r_2, t) = \langle E_1(r_1, t_1)\, E_j(r_2, t_2) \rangle \ .$$

He then observes that these functions satisfy the two wave equations

$$(10a) \qquad \frac{1}{c^2} \frac{\partial^2 \mathscr{E}_{ij}}{\partial t^2} = \nabla_1^2 \mathscr{E}_{ij} \ ,$$

$$(10b) \qquad \frac{1}{c^2} \frac{\partial^2 \mathscr{E}_{ij}}{\partial t^2} = \nabla_2^2 \mathscr{E}_{ij} \ .$$

Our stipulation of spatial stationarity enables us to write, in place of equations (10a) and (10b), the single equation

$$(11) \qquad \frac{1}{c^2} \frac{\partial^2 \mathscr{E}_{ij}}{\partial t^2} = \nabla^2 \mathscr{E}_{ij} \ .$$

This may be shown in an elementary way from the definition $\mathscr{E}_{ij}$ and the assumption that the electric field components individually satisfy a wave equation of the form of equation (11). From the nature of stationary and isotropic fluctuations, it follows that the $\mathscr{E}_{ij}$ must be even functions of the arguments r and t; thus, a general even solution of (11) may be composed from the elementary solutions

$$(12) \qquad \exp{[i\,k \cdot r]} \cos kct$$

by the integral

$$(13) \qquad \mathscr{E}_{ij}(r, t) = \int \exp{[i k \cdot r]} \cos kct f_{ij}(k)\, \mathrm{d}^3 k \ ,$$

providing that

$$(14) \qquad f_{ij}(k) = f_{ij}(-k) \ .$$

We now examine the case of simultaneous correlations (*i.e.*, $t = 0$). Equation (13) becomes

(15)
$$\mathscr{E}_{ij}(\boldsymbol{r}, 0) = \int \exp [i\boldsymbol{k} \cdot \boldsymbol{r}] f_{ij}(\boldsymbol{k}) \, \mathrm{d}^3 k \ .$$

3. – Boundary conditions and the Planck radiation law.

The functions represented are the spatial cross-correlation functions of the vector field components E_i. They are analogous to the velocity correlation field studied in the theory of homogeneous turbulence. In particular, they are analogous to the correlation functions of velocity in isotropic turbulence. The further restriction to the turbulence of an incompressible fluid, guaranteed by the condition

(16)
$$\operatorname{div} \boldsymbol{v} = \boldsymbol{0} \ ,$$

has its exact equivalent in our vacuum condition, equation (3):

(17)
$$\operatorname{div} \boldsymbol{E} = 0 \ .$$

Under these conditions of incompressibility (here, equation (17)) and isotropicity (equation (5)) it has been shown (by BATCHELOR [2], ROBERTSON [3], and KAMPÉ DE FÉRIET [4]) that $f_{ij}(\boldsymbol{k})$ takes the special form

(18)
$$f_{ij}(\boldsymbol{k}) = A(k)(k^2 \delta_{ij} - k_i k_j) \ .$$

We have now only to determine the scalar function $A(k)$. This can be done as follows: Substitute (18) into equation (13) for the special case in which $\boldsymbol{r} = 0$. There results, after an integration over solid angle,

(19)
$$\mathscr{E}_{ij}(0, t) = \left(\frac{8\pi}{3} \delta_{ij}\right) \int_0^\infty k^4 A(k) \cos kct \, \mathrm{d}k \ .$$

Now according to the Wiener-Khintchine theorem the time cross-correlation functions (in our case $\mathscr{E}_{ij}(0, t)$) if stationary random functions are derivable

[2] G. K. BATCHELOR: *The Theory of Homogeneous Turbulence* (Cambridge, 1953).

[3] H. P. ROBERTSON: *Proc. Camb. Phil. Soc.*, **36**, 209 (1940).

[4] J. KAMPÉ DE FÉRIET: *Introduction to the statistical theory of turbulence-correlation and spectrum*, Lecture Series no. 8, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland (1951).

from their cross-spectral densities by the formula

$$(20) \qquad \mathscr{E}_{ij}(0, t) = \int_0^\infty \Phi_{ij}(k) \cos kct \, dk \, .$$

It is thus easily seen, by comparison with (19), that $\Phi_{ij}(k)$ and $A(k)$ are related by

$$(21) \qquad \Phi_{ij}(k) = \frac{8\pi}{3} \delta_{ij} k^4 A(k) \, .$$

The cross-spectral densities are simply related to the Planck distribution function for blackbody radiation since the total energy density, U, is the summation of the energy densities at all wave numbers. Thus

$$(22) \qquad U = \frac{4\hbar c}{3\pi} \int_0^\infty \frac{k^3 \, dk}{\exp\left[\alpha k\right] - 1} \, , \qquad\qquad \alpha = \frac{\hbar c}{kT} \, .$$

Equations (20) and (6) give

$$(23) \qquad \mathscr{E}_{ij}(0, 0) = \langle E_i E_j \rangle = \frac{4\pi}{3} \delta_{ij} U = \int_0^\infty \Phi_{ij}(k) \, dk \, .$$

Hence we obtain for $\Phi_{ij}(k)$ the expression

$$(24) \qquad \Phi_{ij}(k) = \left(\frac{4}{3}\right)^2 \hbar c \delta_{ij} \frac{k^3}{\exp\left[\alpha k\right] - 1} \, .$$

Using the relation (21) we obtain for $A(k)$

$$(25) \qquad k A(k) = \frac{2\hbar c}{3\pi} \frac{1}{\exp\left[\alpha k\right] - 1} \, .$$

The coherence matrix $\mathscr{E}_{ij}(\mathbf{r}, t)$ can now be obtained by the straightforward use of (13). It will be more instructive, however, to calculate the *time* and the *space* coherence functions separately, and to examine only special forms of the latter called the *lateral* and the *longitudinal* correlations (to be defined later). The time correlation functions are obtained from equation (19), and they prove to be

$$(26) \qquad \mathscr{E}_{ij}(0, t) = -\left(\frac{4}{3}\right)^2 \hbar c \delta_{ij} \frac{1}{c^3} \frac{\partial^3}{\partial t^3} \int_0^\infty \frac{\sin kct \, dk}{\exp\left[\alpha k\right] - 1} \, ,$$

$$(27) \qquad = -\delta_{ij} \frac{8}{9} \hbar c \left(\frac{\pi}{\alpha}\right)^4 L'''(\tau) \, , \qquad\qquad \tau = \frac{\pi c}{\alpha} t \, ,$$

where

(28)
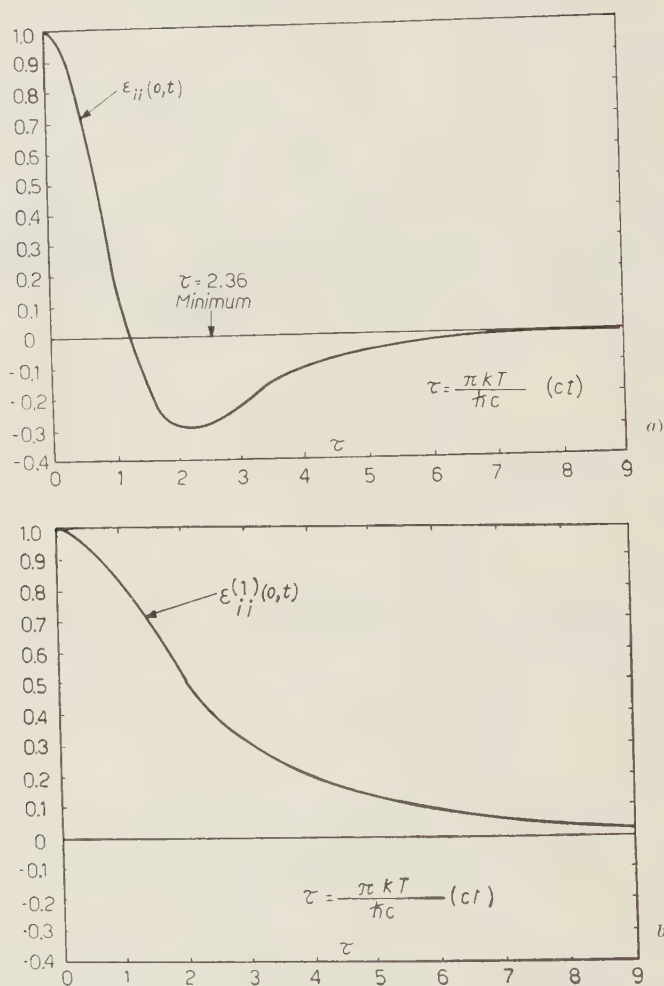$$L(\tau) = \text{ctgh } \tau - \frac{1}{\tau} \; (^*).$$

Fig. 1.

This coherence function (**), conveniently normalized, is displayed in Fig. 1a. For purposes of comparison, the corresponding coherence function for the (rather artificial) case of a one-dimensional pencil (one degree of freedom in

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(*) Called the Langevin function in statistical mechanics.

(**) The author has just learned of some umpublisded studies by G. HULTGREN of the California Institute of Tecnology in which this result, equation (27), is derived.

wave number) of blackbody radiation is easily shown to be

$$(29) \qquad \mathcal{E}_{ii}^{(1)}(0, t) \approx L'(\tau) ,$$

and this function is presented in Fig. 1b.

4. – Spatial coherence.

Because of the complications due to the vector character of the fields, the spatial coherence functions may be more easily interpreted if attention is confined to the *longitudinal* and the *lateral* case, *i.e.*, the cases in which the spatial separation is parallel and perpendicular, respectively, to the direction of a typical autocorrelated field component. The definitions are illustrated in Fig. 2. For brevity, we shall avail ourselves of some of the general formulae of the kinematics of isotropic turbulence of incompressible fluids. One such result is an expression for the (simultaneous) longitudinal correlation function in terms of $A(k)$. It is given (see *e.g.*, PAI ([5]), page 228) by

$$\varepsilon_{ii}^{long}(r,0) = \langle E_i(0,t)E_i(r,t)\rangle$$

$$\varepsilon_{jj}^{lat}(r,0) = \langle E_j(0,t)E_j(r,t)\rangle$$

Fig. 2.

$$(30) \qquad \mathcal{E}_{ii}^{long}(r, 0) = \frac{8\pi}{r^3} \int_0^\infty k^2 A(k) \left[\frac{\sin kr}{kr} - \cos kr \right] dk .$$

It is convenient to rewrite this in the form

$$(31) \qquad \mathcal{E}_{ii}^{long}(r, 0) = \frac{8\pi}{r^3} \left(1 - r\frac{\partial}{\partial r} \right) \int_0^\infty k A(k) \sin kr \, dk .$$

Use of result (25) in this gives

$$(32) \qquad \mathcal{E}_{ii}^{long}(r, 0) = \frac{8hc\pi}{3\alpha} \frac{1}{r^3} \left(1 - r\frac{\partial}{\partial r} \right) L\left(\frac{\pi}{\alpha}r \right) ,$$
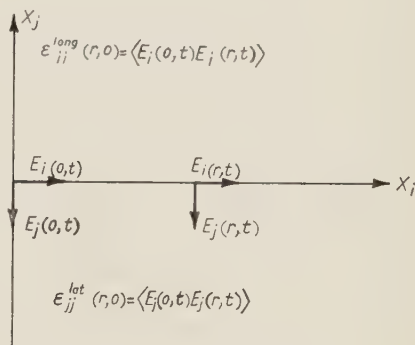

([5]) SHIH-I PAI: *Viscous Flow Theory* - II: *Turbulent Flow* (New York, 1957), p. 228.

23 - *Il Nuovo Cimento*.

where $L(x)$ is the Langevin function defined previously. This longitudinal coherence function is shown in Fig. 3. It is a standard result (*) of the theory
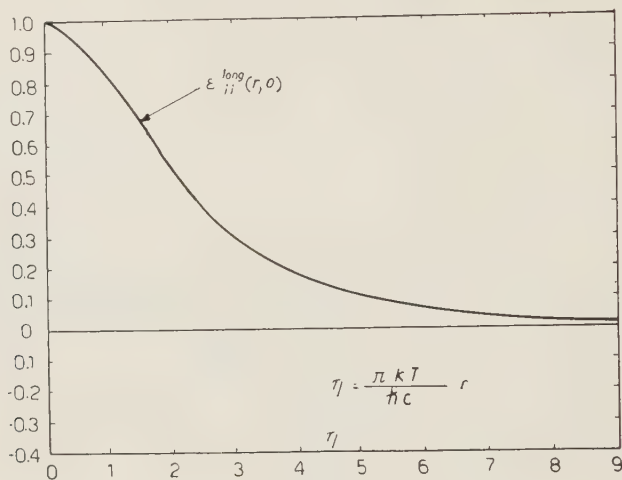

Fig. 3.

of the kinematics of incompressible isotropic turbulence that the longitudinal and the lateral correlations are related by

$$(33) \qquad \mathscr{E}_{ii}^{\text{lat}}(r,\, 0) = \left(1 + \frac{r}{2}\, \frac{\partial}{\partial r}\right) \mathscr{E}_{ii}^{\text{long}}(r,\, 0).$$

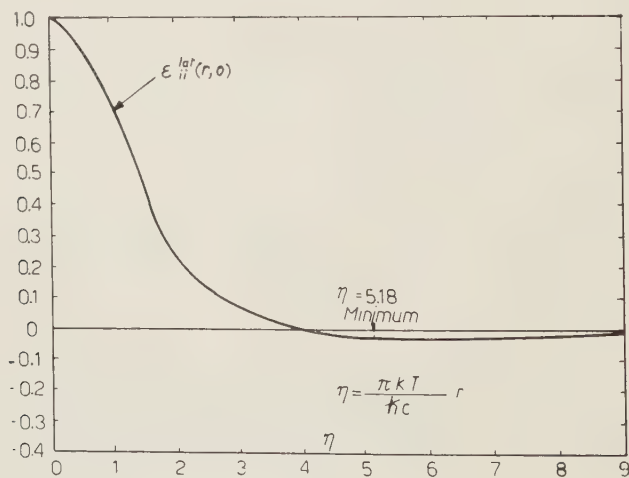The lateral coherence function is shown in Fig. 4.

Fig. 4.

(*) Cf., *e.g.*, PAI: *op. cit.*, p. 193.

If it is desired to include the time (difference) dependence into the longitudinal or lateral coherence functions, one may simply replace $A(k)$ by $A(k) \cos kct$ in equation (29). By virtue of an elementary property of *sine* transforms there results

$$(34) \qquad \mathscr{E}_{ii}^{\text{long}}(r,\, t) = \frac{1}{2} [\mathscr{E}_{ii}^{\text{long}}(r + ct,\, 0) + \mathscr{E}_{ii}^{\text{long}}(r - ct,\, 0)] \,,$$

and similarly for the lateral coherence function.

The results obtained here for the electric field are applicable without change to the magnetic field in view of the assumption $\langle E^2 \rangle = \langle H^2 \rangle$ and since $\operatorname{div} \boldsymbol{H} = 0$ is always satisfied.

If coherence functions for the vector potential are developed, cross-correlations between electric and magnetic fields may readily be expressed. This would be, in fact, a more satisfactory procedure from a general point of view, but the additional information obtained from it is of little value. One point however, seems worthy of mention, namely, that the time coherence of the vector potential is identical to that obtained for the electric field in the fictitious example of one-dimensional cavity radiation, Fig. 1b. This can be seen as follows: define

$$(35) \qquad \mathscr{A}_{ij}(0,\, t) = \langle A_j(0,\, t_1) A_j(0,\, t_2) \rangle \,, \qquad\qquad t = t_2 - t_1.$$

Differentiating and using the stationarity of the bracketed function, we find that

$$(36) \qquad \mathscr{A}_{ij}''(0,\, t) = - \langle A_i'(0,\, t_1) A_j'(0,\, t_2) \rangle \,.$$

We are free to choose Coulomb gauge (in vacuo) *i.e.*, $\operatorname{div} \boldsymbol{A} = 0$ and $\varphi = 0$, so that

$$(38) \qquad E_i(0,\, t) = - \frac{1}{c} A_i'(0,\, t) \,.$$

Substituting this into (36) gives

$$(39) \qquad \mathscr{A}_{ij}''(0,\, t) = - c^2 \langle E_i(0,\, t_1) E_j(0,\, t_2) \rangle = - c^2 \ddot{\mathscr{E}}_{ij}(0,\, t) \,.$$

Referring to equation (27) shows finally that

$$(40) \qquad \mathscr{A}_{ij}(0,\, t) = \delta_{ij} \frac{8}{9} \hbar c^3 \left(\frac{\pi}{\alpha} \right)^4 L'(\tau) \approx \mathscr{E}_{ij}^{(1)}(0,\, t) \,.$$

The vanishing divergence of $\boldsymbol{A}$ in this gauge makes possible the same development of spatial coherence functions for $\boldsymbol{A}$ as was employed previously for $\boldsymbol{E}$ (or $\boldsymbol{H}$), but we shall not develop the potential coherence functions further.

APPENDIX

Explicit formulas for coherence functions shown in the figures (subscript is figure number):

$$f_{1a}(x) = 15 \operatorname{cosech}^2 x \, [3 \operatorname{ctgh}^2 x - 1] - \frac{45}{x^4},$$

$$f_{1b}(x) = -3 \operatorname{cosech}^2 x + \frac{3}{x^2},$$

$$f_3(x) = \frac{45}{2x^3} \left[\operatorname{ctgh} x + x \operatorname{cosech}^2 x - \frac{2}{x} \right],$$

$$f_4(x) = \frac{45}{2x} \left[\frac{2}{x^3} - \operatorname{cosech}^2 x \cdot \operatorname{ctgh} x - \frac{1}{2x} \operatorname{cosech}^2 x - \frac{1}{2x^2} \operatorname{ctgh} x \right].$$

Since all these functions take the form of small differences of large numbers near $x = 0$, it is convenient to employ the following series expansions around the origin:

$$f_{1a}(x) = 1 + \frac{15}{2} \sum_{n=3}^{\infty} (-1)^n \, \frac{2^{2n} B_{2n-1}}{(2n)(2n-4)!} \, x^{2n-4},$$

$$f_{1b}(x) = 1 + 3 \sum_{n=2}^{\infty} (-1)^{n-1} \, \frac{2^{2n} B_{2n-1}}{(2n)(2n-2)!} \, x^{2n-2},$$

$$f_3(x) = 1 + \frac{45}{2} \sum_{n=3}^{\infty} (-1)^n \, \frac{2^{2n} 2(n-1) B_{2n-1}}{(2n)!} \, x^{2n-4},$$

$$f_4(x) = 1 + \frac{45}{2} \sum_{n=3}^{\infty} (-1)^n \, \frac{2^{2n} 2(n-1)^2 B_{2n-1}}{(2n)!} \, x^{2n-4}.$$

The B_n are the Bernoulli numbers.

RIASSUNTO (*)

Si presenta un calcolo esplicito delle funzioni di correlazione dello spazio e del tempo delle componenti del campo elettrico nella radiazione del corpo nero. Queste funzioni sono determinate dalla statistica di un gas fotonico poichè la legge di distribuzione di Planck è essenzialmente lo spettro di potenza delle fluttuazioni di campo. Si sviluppano le funzioni spaziali di coerenza in stretta analogia con la teoria della turbolenza isotropica di un fluido incompressibile.

(*) *Traduzione a cura della Redazione.*

The Two-Body Problem in the Extreme Relativistic Limit (*).

K. Baumann and W. Thirring

Institute for Theoretical Physics, University of Vienna - Vienna

Summary. — In view of the composite model for elementary particles the two-body prob'em is studied in the extreme relativistic limit. Spinless particle-antiparticle bound states are investigated for a single massless self-interacting Weyl field. High momenta are suppressed in a Lorentz invariant manner by introducing a heavy auxiliary boson. A Bethe-Salpeter equation is set up in the ladder approximation. The consequences of the continuity equation are studied in the same approximation and are shown to be fulfilled automatically only for scalar bound states. A new method is presented for solving the Bethe-Salpeter equation. A spectral ansatz is made for the wave function which leads to a one-dimensional integral equation for the weight function when the rest mass of the bound state is neglected. This integral equation is solved in the Fredholm approximation for a somewhat modified cut-off prescription in the scalar case. With the same restriction, the eigenvalue is found also in the pseudoscalar case. It is shown that the method of solution can be extended to any order in the rest mass of the bound state.

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1. – Introduction.

Many years ago, Fermi and Yang [1] raised the question if the fundamental strong interaction could be a contact interaction between nucleons. They suggested that pions might be considered as composite systems, consisting of a nucleon and an antinucleon bound by this contact force. If so, the bound state problem is an extreme-relativistic one, and should be treated

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[1] E. Fermi and C. N. Yang: *Phys. Rev.*, **76**, 1739 (1948).

by a Bethe-Salpeter equation. This has been done by a number of authors, we only mention Z. MAKI [2] and J. V. POLUBARINOV [3]. These authors consider a Bethe-Salpeter equation in the chain approximation. In this case the problem of solving the integral equation for $\chi(x, y)$ reduces to a determination of $\chi(x, x)$. Generally the introduction of some cut-off device is necessary to get finite expressions.

One of us has tried to reduce all strongly interacting elementary particles to three fundamental massless Weyl fields [1]. There for a particle-antiparticle bound state one finds $\chi(x,x) = 0$. Therefore one has to smear out the interaction such as to make the integral equation a non-trivial one. This we shall do by introduction of an intermediate vector boson which nowadays is quite popular.

So far Bethe-Salpeter equations in the extreme-relativistic case have been solved only when the mass of the intermediate boson is zero [5-8]. Here we are faced with the opposite case of a boson whose rest mass is large compared with the other masses in the problem. We shall present a method of getting a solution of such an equation as a power series in $M^{-2}q^2$ and $M^{-2}q^\nu \partial_\nu$, where q^ν is the energy-momentum of the bound state, and M is the rest mass of the heavy boson.

2. – Reflection properties.

The theory we consider is invariant under the operation $PC \equiv R$. The fundamental two component spinor thereby transforms like

$$(1) \qquad \begin{cases} R\psi(x) R^{-1} = g\psi^+(x_r) , \\ g = \sigma_2 , \qquad (x_r^0, \boldsymbol{x}_r) = (x^0, -\boldsymbol{x}) . \end{cases}$$

We suppose that there exists a spinless bound state which we would like to identify wity the pion [9]. Following ZIMMERMANN [10], then, we can define

[2] Z. MAKI: *Progr. Theor. Phys.*, **16**, 667 (1956).

[3] J. V. POLUBARINOV: *Nucl. Phys.*, **8**, 444 (1958).

[4] W. THIRRING: *Nucl. Phys.*, **10**, 97 (1959); **14**, 565 (1960). Also the equation investigated by HEISENBERG *et al.* corresponds to a description of the nucleons by Weyl fields.

[5] J. S. GOLDSTEIN: *Phys. Rev.*, **91**, 1516 (1953).

[6] G. C. WICK: *Phys. Rev.*, **96**, 1124 (1954).

[7] R. E. CUTKOSKY: *Phys. Rev.*, **96**, 1135 (1954).

[8] S. OKUBO and D. FELDMAN: *Phys. Rev.*, **117**, 279 (1960).

[9] For simplicity, we neglect isotopic spin degrees of freedom.

[10] W. ZIMMERMANN: *Nuovo Cimento*, **10**, 597 (1958).

a pion field operator $\varphi(x)$. We must have

(2)
$$R\varphi(x)R^{-1} = -\varphi(r_r),$$

since empirically the pion has odd parity and even charge parity. In what follows we shall treat the more general case

(2')
$$R\varphi(x)R^{-1} = \pm\varphi(x_r).$$

Furthermore, there exists an antilinear operator $TCP = \Theta$ which leaves the theory invariant. We have

(3)
$$\Theta\psi(x)\Theta^{-1} = \psi^{+}(-x),$$

(4)
$$\Theta\varphi(x)\Theta^{-1} = \varphi(-x).$$

The properties

(5)
$$\tau_{\alpha\beta}(x, y, z) = T\langle\psi_\alpha(x)\,\psi_\beta^\dagger(y)\,\varphi(z)\rangle_0,$$

resulting from (1) to (4) are readily examined. R-invariance yields

(6)
$$\tau(x, y, z) = \mp g\tau^T(y_r, x_r, z_r)g,$$

the two signs corresponding to the two possible signs in (2'). Θ-invariance leads to

(7)
$$\tau(x, y, z) = -\tau(-x, -y, -z).$$

The Bethe-Salpeter wave function of a pion with energy-momentum q,

(8)
$$\chi_q(x, y) = \langle 0|\psi(x)\,\psi^{+}(y)|q\rangle$$

is connected with $\tau(x, y, z)$ by

(9)
$$\chi_q(x, y) = (2\pi)^{-\frac{1}{2}}\int dz \exp\left[-iqz\right](\Box_z - \mu^2) + \tau(x, y, z).$$

If we separate the center of mass motion,

(10)
$$\chi_q(x, y) = \exp\left[-i\tfrac{1}{2}q(x + y)\right]\varphi_q(x - y),$$

we find from (6), (7)

$$(11) \qquad \varphi_q(x) = \pm\, g\, \varphi_{qr}^T(-x_r) \,,$$

$$(12) \qquad \varphi_q(x) = -\, \varphi_{-q}(-x) \,.$$

The last two equations are equivalent with

$$(13) \qquad \varphi_q(x) = (\sigma x)\, f(x^2, (qx)^2) + (\sigma q)(qx))\, g(x^2, (qx)^2)\,, \qquad \text{(scalar)}$$

$$(14) \qquad \varphi_q(x) = (\sigma q)\, f(x^2, (qx)^2) + (\sigma x)(qx))\, g(x^2, (qx)^2)\,, \quad \text{(pseudoscalar)}$$

3. – Bethe-Salpeter equation.

We shall get approximate expressions for the functions f, g by solving a Bethe-Salpeter equation in the chain-approximation. In order to get finite answers, it is necessary to introduce a momentum (exchange) cut-off M. This we do in a Lorentz invariant manner by introducing a heavy auxiliary boson of mass M, thus writing down a ladder approximation instead of a chain approximation.

Thus, the field equation is ([11])

$$(15) \qquad (\bar\sigma,\, p - \lambda A)\psi = 0 \,, \qquad\qquad p = i\partial \,,$$

which we use to get

$$p_\mu p'_\nu \bar\sigma^\mu \langle 0 \mid T\,\psi(x)\,\psi^+(x') \mid q \rangle \bar\sigma^\nu = \lambda^2\, \bar\sigma^\mu \langle 0 \mid T\,\psi(x)\,\psi^+(x')\, A_\mu(x)\, A_\nu(x') \mid q \rangle \bar\sigma^\nu \simeq$$

$$\simeq \lambda^2 \langle T A_\mu(x)\, A_\nu(x') \rangle_0\, \bar\sigma^\mu \langle 0 \mid T\,\psi(x)\,\psi^+(x') \mid q \rangle \bar\sigma^\nu \,,$$

or ([12])

$$(16) \qquad (p - \tfrac{1}{2}q)_\mu (p + \tfrac{1}{2}q)_\nu\, \bar\sigma^\mu \varphi(x)\bar\sigma^\nu = -\, 2i\lambda^2\, \bar\varphi(x)\, \varDelta_f(x,\, M^2) \,,$$

([11]) The reader is reminded of the types of σ's:

$$\sigma_\lambda \sim (1, \boldsymbol\sigma), \qquad \sigma^\lambda \sim (1, -\boldsymbol\sigma) \,,$$

$$\bar\sigma_\lambda \sim (1, -\boldsymbol\sigma), \qquad \bar\sigma^\lambda \sim (1, \boldsymbol\sigma) \,.$$

([12]) It should be noticed that we have derived a Bethe-Salpeter equation for two massless nucleons. However, vanishing of the physical nucleon mass is *not* a consequence of the zero bare nucleon mass in this theory, except in perturbation theory, as has been pointed out before: W. THIRRING: *Nucl. Phys.*, **14**, 565 (1960). Our results should be understood as a zero order approximation in the nucleon mass, in a sense which will be shown below.

where

(17)
$$\overline{\varphi}(x) = - 2\overline{\sigma}_\mu \varphi(x)\overline{\sigma}^\mu \ .$$

That is, by virtue of

(18)
$$\sigma_\mu \overline{\sigma}_\nu \sigma^\mu = - 2\sigma_\nu$$

we get $\overline{\varphi}(x)$ from $\varphi(x)$ by replacing all σ's in (13) and (14) by $\overline{\sigma}$'s. The field equation (15) has still another consequence, namely

(19)
$$p_\mu : \psi^+(x)\overline{\sigma}^\mu \psi(x) := 0 \ .$$

Hence

(20)
$$p_\mu \langle 0 | : \psi^+(x)\overline{\sigma}^\mu \psi(x) : | q \rangle = - q_\mu \exp[-iqx] \, \mathrm{Sp}\,[\varphi_q(0)\overline{\sigma}^\mu] =$$
$$= - \exp[-iqx] \, \mathrm{Sp}\,[\varphi_q(0)(\overline{\sigma}q)] = 0 \ .$$

This is identically satisfied by any scalar wave function (13). It is satisfied by a wave function of the type (14) provided $q^2 = 0$.

One can derive a more general condition for the Bethe-Salpeter function from (15). Rewrite (15) as

(21)
$$\psi(x) = \psi_{(\mathrm{in})}(x) + \int \mathrm{d}x' \, S_R(x - x')\overline{\sigma}^\lambda A_\lambda(x')\,\psi(x') \ .$$

Now $[\psi(x), \psi^+(x)]$ can be expressed in either of the following ways:

$$\left[\psi^{(\mathrm{in})}(x) + \int \mathrm{d}x' \, S_R(x - x')\sigma^\lambda A_\lambda(x')\,\psi(x') \,, \ \psi^+(x) \right]$$

and

$$\left[\psi(x), \ \psi^{(\mathrm{in})+}(x) + \int \mathrm{d}x' \, \psi^+(x')\overline{\sigma}_\lambda A^\lambda(x') \, S_R^+(x - x') \right] \ .$$

The same approximation which led to eq. (16) yields

(22)
$$\tfrac{1}{2} \langle 0 | [\psi(x), \psi^+(x)] A^\nu(x) | q \rangle = \int \mathrm{d}x' \, \varDelta^{(+)}(x' - x) \, S_R(x - x')\overline{\sigma}^\nu \chi_q(x', x) =$$
$$= \int \mathrm{d}x' \, \varDelta^{(+)}(x' - x) \, \chi_q(x, x')\overline{\sigma}^\nu S_R^+(x - x') \,,$$

which is certainly fulfilled if

(23)
$$S_R(x - x')\overline{\sigma}^\nu \chi_q(x', x) = \chi_q(x, x')\overline{\sigma}^\nu S_R^+(x - x') \ .$$

Now $S_R(x)$ is proportional to (σx), furthermore

$$S_R^+(x) = - i(\sigma \partial) \Delta_R(x) = - S_R(x) ,$$

so that (25) and hence (24) is automatically fulfilled by the scalar wave function.

In the pseudoscalar case the condition (20), (22) can be checked only after one has solved the Bethe-Salpeter equation.

4. – Scalar solution.

We seek a solution of eq. (16) in the limit of $q^2 \ll M$, $m^2 \ll M^2$ (m is the nucleon mass), and try the following spectral ansatz:

$$(24) \qquad \varphi_q(x) = (\sigma p) \int da \, \{\varrho_1(a) + q^2 \varrho_2(a) + (qp)^2 \varrho_3(a) + ...\} \Delta_f(x, a) +$$

$$+ (\sigma q)(qp) \int da \{\overline{\varrho}_1(a) + ...\} \Delta_f(x, a) .$$

This is in agreement with (13), and furthermore, it is of the general form derived by JOST and LEHMANN [13] for matrix elements of the type $T\langle \alpha | A(x) B(y) | \beta \rangle$. Hence we are safe that our solution will not violate any analyticity requirements.

Introducing (24) into (16) and keeping only the leading terms in q, we get the equation

$$(25) \qquad (\sigma p) p^2 \int da \, \varrho_1(a) \Delta_f(x, a) = (\sigma p) \int dc \int da \, \varrho_1(a) \, g_0(a, c) c \, \Delta_f(x, c) ,$$

where $g_0(a, c)$ is defined by

$$(26) \qquad 2i\lambda^2 \Delta_f(x, M^2) p_\nu \Delta_f'(x, a) = p_\nu \int dc \, g_0(a, c) c \, \Delta_f(x, c)$$

and is calculated in the Appendix. The r.h.s. of (25) can be rewritten by the use of

$$(27) \qquad c \, \Delta_f(x, c) = p^2 \left\{ \Delta_f(x, c) - \int dc' \, \delta(c') \, \Delta_f(x, c') \right\} ,$$

[13] R. JOST and H. LEHMANN: Nuovo Cimento, 5, 1598 (1957).

so that we finally get

(28) $\quad (\sigma p) p^2 \int dc\, \varDelta_f(x,\, c)\, \varrho_1(c) =$

$$= - (\sigma p) p^2 \int dc\, \varDelta_f(x,\, c)\, \left\{ \int da\, \varrho_1(a)\, g(a,\, c) - \delta(c) \int da \int db\, \varrho_1(a)\, g(a,\, b) \right\}.$$

The following integral equation is sufficient for (28) to be fulfilled:

(29) $$\varrho_1(c) = \int da\, \varrho_1(a)\, g(a,\, c) - \delta(c) \int da \int db\, \varrho_1(a)\, g(a,\, b)\, .$$

It is an immediate consequence of (29) that

(30) $$\int \varrho_1(c)\, \check{c}c = 0\, .$$

If we put

(31) $$\varrho(c) = \delta(c) + \sigma(c)$$

we get from (29) the pair of equations

(30a) $$\sigma(c) = g_0(0,\, c) + \int da\, \sigma(a)\, g_0(a,\, c)\, ,$$

(32b) $$\int \sigma(c)\, dc = -1\, .$$

The slow decrease at infinity of

(33) $\quad g_0(a,\, c) = (\lambda/4\pi)^2\, c^{-3}\, [(c + a - M^2)^2 - 4ac]^{\frac{1}{2}}\, (c + a - M)\, \theta(\sqrt{\check{c}} - \sqrt{a} - M)\, ,$

(see Appendix) keeps us from getting a finite $\int \sigma(c)\, dc$ for the first Fredholm approximation of $\sigma(c)$. The solution of this difficulty will be given in a forthcoming paper. We shall now examine the solution of the Bethe-Salpeter equation corresponding to a somewhat different cut-off prescription. On the r.h.s. of (16) $\varDelta_f(r,\, M^2)$ will be replaced by $-\partial \varDelta_f(r,\, M^2)/\partial M^2$. This changes (33) into

(33') $$g_0(a,\, c) = \frac{\lambda^2}{8\pi^2}\, \frac{1}{c^3}\, \frac{(c + a - M^2)^2 - 2ac}{[(c + a - M^2)^2 - 4ac]^{\frac{1}{2}}}\, \theta(\sqrt{\check{c}} - \sqrt{a} - M)\, .$$

This, in a first approximation, leads to

(34) $$\sigma(c) = (\lambda^2/8\pi^2)\, c^{-3}(c - M^2)\, \theta(c - M^2)\, ,$$

which, combined with (32b), yields

$$(35) \qquad (\lambda/4\pi)^2 = -M^2 .$$

Hence we get a scalar solution of the Bethe-Salpeter equation for an imaginary value of the coupling constant only. Of course, reality of λ means hermiticity of the interaction as long as we work with eq. (16). But introducing $-\partial \Delta_f(x, M^2)/\partial M^2$ as a boson propagator corresponds to having ghost states anyhow, and we might as well replace λ^2 by $-\lambda^2$ and at the same time consider $+\partial \Delta_f(x, M^2)/\partial M^2$ as the boson propagator. Hence the reality properties of λ do not mean very much in these preliminary calculations.

It should be pointed out that the method of solution presented here may be extended to any order of q^2. Thus, putting the integrands of

$$(\sigma p)p^2 \int dc\, \Delta_f(x, c) ,$$

$$(\sigma p)q^2 p^2 \int dc\, \Delta_f(x, c) ,$$

$$(\sigma p)(qp)^2 p^2 \int dc\, \Delta_f(x, c) ,$$

$$(\sigma p)(qp)p^2 \int dc\, \Delta_f(x, c)$$

on both sides equal, we get four integral equations for the functions $\varrho_1, \varrho_2, \varrho_3, \overline{\varrho}_1$ in (24).

5. – Pseudoscalar solution.

We now restrict ourselves on a discussion of the equation

$$(36) \qquad (\sigma, p + \tfrac{1}{2}q)\varphi(x)(\overline{\sigma}, \overleftarrow{p} - \tfrac{1}{2}q) = 2i\lambda^2 \, \overline{\varphi}(x)\, \partial \Delta_f(x, M^2)/\partial M^2 .$$

In lowest non-vanishing order in q the spectral ansatz is

$$(37) \qquad \varphi_q(x) = (\sigma q)\int da\, \varrho(a)\, \Delta_f(x, a) + (\sigma p)(qp)\int da\, \overline{\varrho}(a)\, \Delta_f(x, a) .$$

By the use of (27), then, the l.h.s. of (36) becomes

$$(38) \qquad (\overline{\sigma} p)(qp)p^2 \int dc\, \Delta_f(x, c) \left\{ 2\frac{\varrho(c)}{c} - 2\, \delta(c)\int da\, \frac{\varrho(a)}{a} + \overline{\varrho}(c) \right\} -$$
$$- (\overline{\sigma} q)p^2 \int dc\, \Delta_f(x, c)\, \varrho(c) ;$$

whereas for the r.h.s. using the combination formulae derived in the Appendix we get

$$(39) \qquad (\bar{\sigma}q)\int dc\, da\, c\varDelta_f(x,\, c)\{\varrho(a)\, g_0(a,\, c) + \bar{\varrho}(a)\, g_2(a,\, c)\} +$$

$$+ (\bar{\sigma}p)(qp)\int dc\, da\, c\varDelta_f(x,\, c)\, \bar{\varrho}(a)\, g_1(a,\, c) .$$

Here

$$(40) \quad \begin{cases} g_0(a,\, c) = \dfrac{\lambda^2}{8\pi^2}\, \dfrac{c+a-M^2}{c^2}\, [(c+a-M^2)^2 - 4ac]^{-\frac{1}{2}}\theta(\sqrt{c} - \sqrt{a} - M), \\[2mm] g_1(a,\, c) = \dfrac{\lambda^2}{8\pi^2}\, \dfrac{c+a-M^2}{c^4}\, \dfrac{(c+a-M^2)^2 - 3ac}{[(c+a-M^2)^2 - 4ac]^{\frac{3}{2}}}\, \theta(\sqrt{c} - \sqrt{a} - M), \\[2mm] g_2(a,\, c) = -\dfrac{\lambda^2}{8\pi^2}\, \dfrac{c+a-M^2}{4c^3}\, [(c+a-M^2)^2 - 4ac_1^{-\frac{1}{2}}\theta(\sqrt{c} - \sqrt{a} - M) . \end{cases}$$

Again using (27) we are led to the pair of integral equations

$$(41a) \qquad \varrho(c) = \int da\, (\varrho(a)\, g_0(a,\, c) + \bar{\varrho}(a)\, g_2(a,\, c)) -$$

$$- \delta(c)\int da\, db\, (\varrho(a)\, g_0(a,\, b) + \bar{\varrho}(a)_2 g_2(a,\, b)),$$

$$(41b) \qquad \bar{\varrho}(c) = -2\, \frac{\varrho(c)}{c} - \int \bar{\varrho}(a)\, g_1(a,\, c)\, da\, + \delta(c)\int db\, \left\{ 2\, \frac{\varrho(b)}{b} + \int da\, \bar{\varrho}(a)\, g_1(a,\, b) \right\} .$$

We put

$$(42) \qquad\qquad\qquad \varrho(c) = \delta(c) + \sigma(c),$$

$$(43) \qquad\qquad\qquad \bar{\varrho}(c) = 2\, \delta'(c) + \alpha\delta(c) + \bar{\sigma}(c)$$

and get the following equations for σ, $\bar{\sigma}$:

$$(44) \qquad\qquad\qquad \int dc\, \sigma(c) = -1,$$

$$(45) \qquad\qquad\qquad \int dc\, \bar{\sigma}(c) = -\alpha,$$

$$(46) \qquad \sigma(c) = g_0(0,\, c) - 2g_2'(0,\, c) + \alpha g_2(0,\, c) + \int da\, (\sigma(a)\, g_0(a,\, c) - \bar{\sigma}(a)\, g_2(a,\, c)),$$

$$(47) \qquad \bar{\sigma}(c) = -2\, \frac{\sigma(c)}{c} + 2\, g_1'(0,\, c) - \alpha g_1(0,\, c) - \int da\, \bar{\sigma}(a)\, g_1(a,\, c) .$$

The constant has to be determined such as to make the equations consistent.

To get an estimate of λ let us evaluate $\sigma(c)$ in a first approximation,

$$(48) \qquad \sigma(c) \simeq g_0(0,\, c) - 2 g_2'(0,\, c) = \frac{\lambda^2}{8\pi^2} \left(\frac{1}{c^2} - \frac{M^2}{c^3} \right) \theta(c - M^2) \,.$$

Using this result we get from (44)

$$(49) \qquad \lambda = 4\pi M \,.$$

Thus there is a pseudoscalar bound state for real values of the coupling constant λ.

<div align="center">APPENDIX</div>

The formula

$$(A.1) \qquad \Delta^{(+)}(x,\, a)\, \Delta^{(+)}(x,\, b) = \int dc\, f(a,\, b,\, c)\, \Delta^{(+)}(x,\, c)$$

or equivalently

$$(A.1') \qquad \int dk\, \Delta^{(+)}(k,\, a)\, \Delta^{(+)}(q - k,\, b) = \int dc\, f(a,\, b,\, c)\, \Delta^{(+)}(q,\, c) \,,$$

where

$$(A.2) \qquad f(a,\, b,\, c) = (4\pi)^{-2} c^{-1} [(c + a - b)^2 - 4ac]^{\frac{1}{2}} \theta(\sqrt{c} - \sqrt{a} - \sqrt{b}) \,,$$

can easily be derived [14]. It can also be shown that the same combination formula holds with the same function $f(a,\, b,\, c)$ for Δ_f's replacing the $\Delta^{(+)}$'s.

Next let us derive

$$(A.3) \qquad \Delta^{(+)}_{|\lambda}(x,\, a)\, \Delta^{(+)}(x,\, b) = \int dc\, g(a,\, b,\, c)\, \Delta^{(+)}_{|\lambda}(x,\, c)$$

or

$$(A.3') \qquad \int dk\, k_\lambda\, \Delta^{(+)}(k,\, a)\, \Delta^{(+)}(q - k,\, b) = \int dc\, q_\lambda\, g(a,\, b,\, c)\, \Delta^{(+)}(q,\, c) \,.$$

We notice

$$(A.4) \qquad \begin{cases} q^2 = c \,, \\[4pt] k^2 = a \,, \\[4pt] (q - k)^2 = c + a - 2qk = b \,. \end{cases}$$

[14] W. THIRRING: *Principles of Quantum Electrodynamics* (New Jork, 1958).

Thus, multiplying (A.3') by q^λ we find

(A.5)
$$\begin{cases} \displaystyle\int dc\, g(a, b, c)\, \Delta^{(+)}(q, c) = \int dk\, \frac{(qk)}{q^2}\, \Delta^{(+)}(k, a)\, \Delta^{(+)}(q - k, b) = \\[2mm] \qquad\qquad\qquad\qquad = \displaystyle\int dq\, \frac{c + a - b}{2c}\, f(a, b, c)\, \Delta^{(+)}(q, c)\,, \\[2mm] g(a, b, c) = \dfrac{c + a - b}{2c}\, f(a, b, c)\,. \end{cases}$$

Similarly,

(A.6) $\Delta^{(+)}_{|\lambda\mu}(x, a)\, \Delta^{(+)}(x, b) - \displaystyle\int dc\, \{f_1(a, b, c)\, \Delta^{(+)}_{|\lambda\mu}(x, c) - g_{\lambda\mu} c\, f_2(a, b, c)\, \Delta^{(+)}(x, c)\}\,,$

(A.6') $\displaystyle\int dk\, k_\lambda k_\mu\, \Delta^{(+)}(k, a)\, \Delta^{(+)}(q - k, b) = \int dc\, (q_\lambda q_\mu f_1 + g_{\lambda\mu} c f_2)\, \Delta^{(+)}(q, c)\,.$

Multiplying this by either $g^{\lambda\mu}$ or $q^\lambda q^\mu$ and using (A.4), (A.1) we get

(A.7)
$$\begin{cases} f_1(a, b, c) = \dfrac{(c + a - b)^2 - ac}{3c^2}\, f(a, b, c)\,, \\[3mm] f_2(a, b, c) = \dfrac{4ac - (c + a - b)^2}{12c^2}\, f(a, b, c)\,. \end{cases}$$

Differentiating (A.1), (A.3), and (A.6) with respect to b we get all the formulae we need in the text.

RIASSUNTO (*)

In vista del modello composto per le particelle elementari si studia il problema dei due corpi nell'estremo limite relativistico. Si esaminano gli stati legati particella-antiparticella privi di spin per un semplice campo di Weyl autointeragente privo di masse. Si sopprimono gli impulsi elevati nel modo invariante di Lorentz introducendo un bosone ausiliario pesante. Si stabilisce una equazione di Bethe-Salpeter nell'approssimazione a gradini. Si studiano le conseguenze della equazione di continuità nella stessa approssimazione e si dimostra che vengono automaticamente soddisfatte solo da stati legati scalari. Si presenta un nuovo metodo per risolvere l'equazione di Bethe-Salpeter. Si fa una ipotesi spettrale per la funzione d'onda che porta ad una equazione integrale unidimensionale per la funzione peso se si trascura la massa di riposo dello stato legato. Questa equazione integrale viene risolta nell'approssimazione di Fredholm nel caso scalare per una prescrizione di cut-off un po' modificata. Con la stessa restrizione, si trova l'autovalore anche nel caso pseudoscalare. Si dimostra che in metodo di risoluzione può essere estesa ad ogni ordine della massa di riposo dello stato legato.

(*) *Traduzione a cura della Redazione.*

On a Quadrilinear K^2-π^2 Coupling.

G. COSTA

Istituto di Fisica dell'Università - Bari
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

L. TENAGLIA

Istituto di Fisica dell'Università - Bari

Summary. — We investigate the effect of the direct K-π coupling on the pion-nucleon and kaon-nucleon inelastic scattering. The incident boson is assumed to interact only with the peripheral pions of the nucleon cloud. The inelastic scattering cross section is expressed, by means of the Chew-Low procedure, in terms of the cross section for the K-π interaction, averaged over the energy. The reliability of the approximation and the branching ratios of the possible reactions are discussed for the various cases.

Direct couplings between the K- and π-mesons have been considered by several authors ([1-7]) in trying, on one hand, to account for the observed multiplet structure of the baryons and, on the other hand, to explain the features of some of the dynamical properties of the strange particles.

We shall consider here only «strong» K-π couplings *i.e.* possessing the internal symmetries of the other known strong interactions; assuming the same parity for the two components of the K-doublet we disregard the possibility of a trilinear coupling, discussed in great detail by PAIS ([5]).

([1]) J. SCHWINGER: *Phys. Rev.*, **104**, 1164 (1956).

([2]) P. BUDINI and L. FONDA: *Nuovo Cimento*, **5**, 666 (1957); P. BUDINI, N. DALLA-PORTA and L. FONDA: *Nuovo Cimento*, **6**, 380 (1957).

([3]) S. BARSHAY: *Phys. Rev.*, **109**, 2160 (1958).

([4]) C. CEOLIN, N. DALLAPORTA and L. TAFFARA: *Nuovo Cimento*, **9**, 353 (1958).

([5]) A. PAIS: *Phys. Rev.*, **112**, 624 (1959).

([6]) G. FEINBERG and F. GÜRSEY: *Phys. Rev.*, **114**, 1153 (1959).

([7]) S. BARSAY: *Phys. Rev.*, **110**, 743 (1958).

Direct quadrilinear K^2-π^2 couplings have been considered by different authors; two interaction Lagrangians which are G-invariant [8] can be easily written down

$$(1) \qquad L = \lambda(\overline{K}^+ K^+ + \overline{K}^0 K^0)(\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \, ,$$

$$(1') \qquad L' = \lambda'(\overline{K}\boldsymbol{\tau}\partial_\mu K - \partial_\mu \overline{K}\boldsymbol{\tau}K)(\boldsymbol{\pi}\boldsymbol{\theta}\partial_\mu\boldsymbol{\pi} - \partial_\mu\boldsymbol{\pi}\boldsymbol{\theta}\boldsymbol{\pi}) \, .$$

The first was proposed by BARSHAY [3]; the second is of the same structure of that proposed by FEINBERG and GÜRSEY [6]: obviously it can give rise, unlike the first, to charge exchange processes.

The Lagrangian (1) was considered by BARSHAY in connection with the scattering of K^+ by nucleons [7]; according to a perturbation calculation the s-wave low energy data are fitted with a coupling constant $\lambda^2/4\pi \approx 1 \div 5$.

The effect of the same coupling was also considered in the pion production in K^+-nucleon collisions. A perturbation calculation was performed by CEOLIN, DALLAPORTA and TAFFARA [4]: the ratio of the pion production to the elastic scattering cross-section, with a coupling constant $\lambda^2/4 \approx 2$, seems to be well below the number suggested experimentally.

The present situation about the K^2-π^2 coupling seems to be still inconclusive, and no experimental evidence has been reached so far.

We shall consider here the possibility of detecting a K^2-π^2 interaction, by analyzing some reactions for which such a coupling might be important. The processes we shall consider are the following:

$$(\mathrm{I}) \qquad \begin{cases} \mathrm{K} + \mathcal{N} \to \mathrm{K} + \mathcal{N} + \pi \\ \overline{\mathrm{K}} + \mathcal{N} \to \overline{\mathrm{K}} + \mathcal{N} + \pi \end{cases}$$

$$(\mathrm{II}) \qquad \pi + \mathcal{N} \to \mathrm{K} + \overline{\mathrm{K}} + \mathcal{N}$$

$$(\mathrm{III}) \qquad \overline{\mathrm{K}} + \mathcal{N} \to \mathrm{Y} + \pi + \pi$$

$$(\mathrm{IV}) \qquad \pi + \mathcal{N} \to \mathrm{Y} + \mathrm{K} + \pi$$

In fact, if we think of the nucleon as a baryon core surrounded by a meson cloud, it is reasonable to assume for the above processes, in the hypothesis of a direct K^2-π^2 coupling, that a peripheral interaction between the incident mesons and a virtual meson of the cloud be important [9-11]. Since the number

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[8] T. D. LEE and C. N. YANG: *Nuovo Cimento*, **2**, 749 (1956); P. ROMAN: *Theory of Elementary Particles* (Amsterdam, 1960).

[9] C. GOEBEL: *Phys. Rev. Lett.*, **1**, 337 (1958).

[10] G. F. CHEW and F. E. LOW: *Phys. Rev.*, **113**, 1640 (1959).

[11] P. ROMAN and I. Y. POMERANČUK: *Nucl. Phys.*, **10**, 492 (1959).

of pions in the cloud has been estimated to be approximately one [12], the interaction of the incoming pion or K-meson will reasonably occur with a single pion of the target nucleon. The interaction between the incoming meson and a virtual K-meson should occur with a smaller probability owing to the fact the kaon cloud is confined to a smaller region. We assume that possible successive interactions with pions can be disregarded, for those processes in which the peripheral interaction occurs in the kaon cloud. We disregard also, for all processes, the effects of final state interactions.

On these hypothesis, the peripheral processes involved in the reactions (I) and (IV) are the following:

(a)
$$\begin{cases} K + \pi \rightarrow K + \pi \\ \overline{K} + \pi \rightarrow \overline{K} + \pi \end{cases} \quad \text{(in reactions (I) and (IV)) ,}$$

(b)
$$\pi + \pi \rightarrow K + \overline{K} \quad \text{(in reactions (II)) ,}$$

(c)
$$K + \overline{K} \rightarrow \pi + \pi \quad \text{(in reactions (III)) .}$$

We assume now that the only important contribution to the reactions (I) to (IV) comes from such peripheral interactions. A general reaction can be represented by the diagram [13] in Fig. 1.

We can then use the prescription given by CHEW and LOW [10] for the scattering of particles by unstable targets. Following this prescription, the cross-section for a reaction of the type given above satisfies, at the unphysical limit $\Delta^2 = -\mu_2^2$, the relation [14]

Fig. 1.

(2)
$$\frac{\partial^2 \sigma}{\partial \Delta^2 \, \partial w^2} = \frac{\Gamma^2}{2\pi} \left(\frac{M_2}{M_1} \right) \frac{\mu_2 q_1'}{q_{1L}^2 (\Delta^2 + \mu_2^2)^2} \, \sigma_{\mu\mu}(w) \, .$$

The quantity Γ^2 is essentially the spin average of the squared matrix ele-

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[12] S. FUBINI: *Nuovo Cimento*, **3**, 1425 (1956).

[13] We denote by M_1, M_2 the masses of the target and recoil baryons, by μ_1 and μ_2 the masses of the incident and intermediate meson and by μ_1', μ_2' the masses of the final state mesons.

[14] We use throughout this paper the same symbols of reference [8]: w is the total energy of μ_1', μ_2' in the barycentral system; Δ^2 is the square difference of the momenta of M_1, M_2; $\sigma_{\mu\mu}$ is the cross section for one of the peripheral processes a, b, c; q_{1L} is the laboratory momentum of μ_1; q_1' is the momentum of μ_1 in a frame where the particle μ_2 is at rest; W is the total energy in the overall barycentral system.

ment of the $(M_1 M_2 \mu_2)$ vertex. For the $(NN\pi)$ vertex, it is given by

$$(3) \qquad \Gamma^2 = \alpha f^2 \frac{\Lambda^2}{\mu_2^2},$$

where $f^2 \approx 0.08$; $\alpha = 1$ for a neutral and $\alpha = 2$ for charged pion. The same relation holds for the (YNK) vertex in the hypothesis of odd (YK) relative parity; in this case, assuming $f^2_{KAN} \approx f^2_{K\Sigma N}$, we can take $f^2 \approx 0.1$, in agreement with the K^+-nucleon scattering and photo-production results ([15-17]). Now we have $\alpha = 1$ for a neutral and $\alpha = 2$ for a charged hyperon. In the hypothesis of even (YK) relative parity, the expression (3), is replaced simply by

$$(3') \qquad \Gamma^2 = \alpha g_K^2,$$

where the coupling constants g_K^2 should have a value ranging from 0.06 ([16]) to 0.9 ([15]).

The approximation used consists in an extrapolation of the result (2) to the physical region: this is equivalent to disregard, in the evaluation of the matrix elements for the reactions (I) to (IV), the contributions of processes with more than one intermediate particle ([18]). Perhaps, one can think that, even if this approximation turns out to be valid in those cases in which the intermediate particle is a pion (reactions (I) and (II)), it may be unreliable when the intermediate particle is a K-meson (reactions (II) and (IV)). In fact, in the latter case the pole $\Lambda^2 = -\mu_K^2$ is much further from the physical region. If we exclude a K^2-π^3 vertex ([19]), the other singularities, besides the pole, are located beyond the points $-9\mu_K^2$ for the reactions (I) and (II), and $-(\mu_K + 2\mu_\pi)^2$ for (III) and (IV). Perhaps, in the latter case, they are not far enough from the pole and their contribution to the matrix element might be important.

Chew and Low's conjecture has been applied to different processes, such as pion production by pions and protons ([10,18,20,21]), and associated production in proton-proton collisions ([22]). Its validity seems to be assured by the anal-

([15]) C. Ceolin and L. Taffara: *Nuovo Cimento*, **5**, 435 (1957).

([16]) B. M. McDaniel, A. Silverman, R. R. Wilson and G. Cortellessa: *Phys. Rev.*, **115**, 1039 (1959).

([17]) J. J. Sakurai: *Phys. Rev.*, **113**, 1679 (1959).

([18]) F. Bonsignori and F. Selleri: *Nuovo Cimento*, **15**, 465 (1960).

([19]) *E.g.* a K^2-π^3 interaction occurring via intermediate baryon-antibaryon fields is excluded by space invariance arguments.

([20]) E. Ferrari: *Nuovo Cimento*, **13**, 1285 (1959).

([21]) I. Derado: *Nuovo Cimento*, **15**, 853 (1960); F. Selleri: *Evidence for two π-π resonances*, CERN preprint (1960).

([22]) E. Ferrari: *Nuovo Cimento*, **15**, 652 (1960).

ysis of processes for which there is experimental information about the cross-sections for both the overall and the peripheral reactions [18,20].

From the relation (2) one sees that the total cross-section $\sigma(W)$ could be obtained by integration on the variables w^2 and Δ^2 (for a fixed valued of W) over the allowed region [10], if the dependence on w of the cross-section $\sigma_{\mu\mu}(w)$ were known. As a first approximation we can, however, proceed as in the following: we replace $\sigma_{\mu\mu}(w)$ by a certain averaged value (which depends on W) $\bar{\sigma}_{\mu\mu}(W)$ over the region of integration of the variables w^2 and Δ^2, and perform the integration of the remaining functions in (2). We obtain in this way

$$(4) \qquad \sigma(W) = \frac{\alpha f^2}{2\pi} \left(\frac{M_2}{M_1} \right) \frac{\mu_\pi^2}{q_{1L}^2} R(W) \, \bar{\sigma}_{\mu\mu}(W) \,,$$

where the function $R(W)$ represents the results of such integrations. We report its behaviour in Fig. 2, 3, for some reactions for which the used prescription can be valid not too far from the threshold. For the reactions (IV) the two possible cases of even and odd (KY) parities are considered.
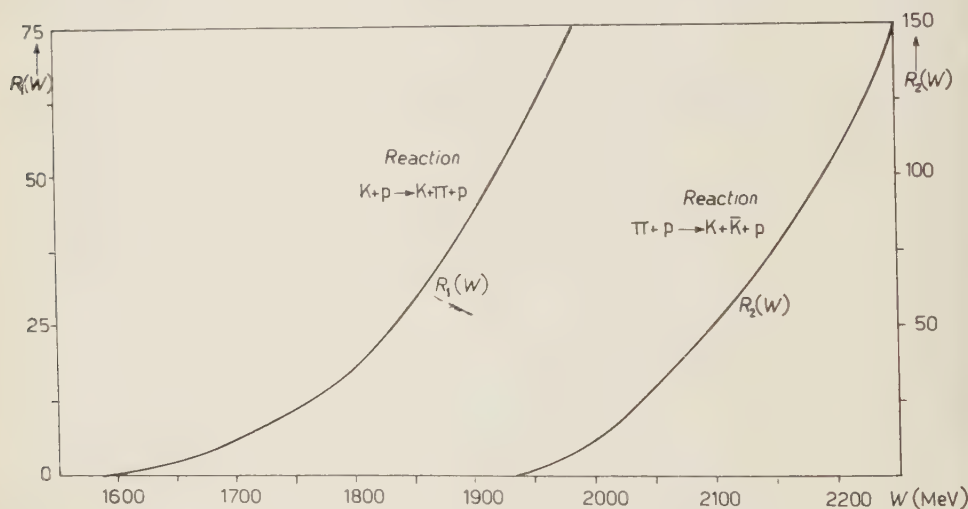

Fig. 2. – Function $R(W)$ for reactions of types I and II.

Let us now examine in more detail the reactions (I)–(IV). In the following tables we specify the different charge states of the overall reactions. For each we indicate the possible peripheral interaction and the corresponding cross-section in terms of the isotopic spin amplitudes. In a few cases (see Tables III and IV) the reaction cannot occur via the mechanism represented in Fig. 1.

by charge or strangeness conservation. In other cases (Tables II and III) the G-invariance [23] applied to the four-meson vertex determines some restrictions in the possible peripheral interactions. Let us first consider the peripheral reaction $\pi + \pi \to$ $\to K + \overline{K}$: it can occur in the isospin eigenstates $t = 0$, $t = 1$, but the relative angular momentum l of the $K\overline{K}$ pair must be even in the isotopic singlet state and odd in the isotopic triplet states.

Correspondingly, if we assume that the K-π interaction is described by one of the two Lagrangians (1), (1′), it is clear that only the $l = 0$, $t = 0$ transitions can be originated by (1′). To single out the contributions of the two Lagrangians (1), (1′) to the different reactions listed in Tables II and III,

Fig. 3. – Function $R(W)$ for reactions of type IV. (The mean value $M_\Sigma \approx 1190$ MeV has been used for the mass of the Σ).

one has simply to retain only one of the two amplitudes b_0, b_1 relative to $t = 0$ $t = 1$. In order to obtain a similar separation for the processes listed in the Tables I and IV (in which the peripheral reactions are of the type $K + \pi \to K + \pi$) we can try to connect the amplitudes a_1, a_3 relative to the two $t = \frac{1}{2}$, $\frac{3}{2}$ eigenstates of the $K\pi$ system with the amplitudes b_0, b_1. The relation is the same as the one connecting the isospin amplitudes of the processes $\pi + \mathcal{N} \to \pi + \mathcal{N}$ and $\pi + \pi \to \mathcal{N} + \overline{\mathcal{N}}$; that is [24]

$$a_1 = \frac{1}{\sqrt{6}} b_0 + b_1 ,$$

$$a_3 = \frac{1}{\sqrt{6}} b_0 - \frac{1}{2} b_1 .$$

The peripheral cross-sections can then be expressed also in Tables I and IV in terms of b_0, b_1. Of course, these amplitudes are to be thought as quite
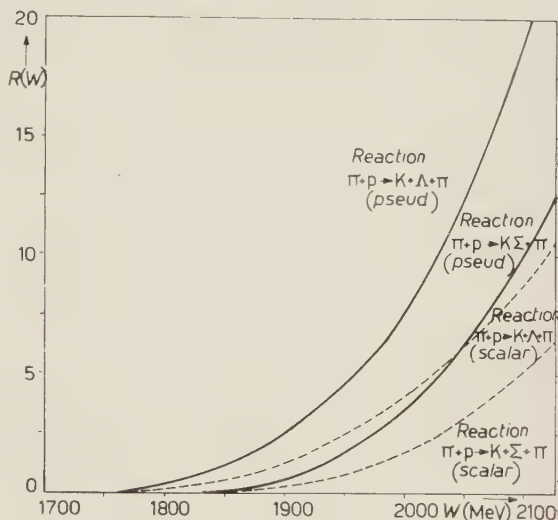
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[23] We remark that the isospin eigenstates of the KK pair are also eigenstates of G: the G-eigenvalue is $(-1)^l$ for the singlet state, and $(-1)^{l+1}$ for the triplet states; l is the relative angular momentum of the two mesons.

[24] W. R. FRAZER and J. R. FULCO: *Phys. Rev.*, **118**, 1603 (1960).

TABLE I (a).

Overall process	Kinetic energy threshold (MeV)	$\left(M_1 M_2/t_2\right)$ vertex	Peripheral reaction	Isospin analysis of the peripheral cross section (b)	Peripheral cross section in terms of the b_0, b_1 amplitudes (c)	Branching ratios (%) for the overall processes in the hypothesis of:			
						K·π dominant state		inter. lagrangian	
						$t=\frac{1}{2}$	$t=\frac{3}{2}$	(1)	(1')
$K^+p \to K^+p\pi^0$	216	$pp\pi^0$	$K^+\pi^0 \to K^+\pi^0$	$\frac{1}{9}\lvert 2a_3+a_1\rvert^2$	$\frac{1}{6}\lvert b_0\rvert^2$	5.5	11	17	0
$\to K^+n\pi^+$	224	$pn\pi^+$	$K^+\pi^+ \to K^+\pi^+$	$\lvert a_3\rvert^2$	$\lvert 1/\sqrt{6}\,b_0-\tfrac{1}{2}b_1\rvert^2$	0	50	33	17
$\to K^0p\pi^+$	226	$pp\pi^0$	$K^+\pi^0 \to K^0\pi^+$	$\frac{2}{9}\lvert a_3-a_1\rvert^2$	$\frac{1}{2}\lvert b_1\rvert^2$	11	5.5	0	17
$K^+n \to K^+n\pi^0$	216	$nn\pi^0$	$K^+\pi^0 \to K^+\pi^0$	$\frac{1}{9}\lvert 2a_3+a_1\rvert^2$	$\frac{1}{6}\lvert b_0\rvert^2$	5.5	11	17	0
$\to K^+p\pi^-$	224	$np\pi^-$	$K^+\pi^- \to K^+\pi^-$	$\frac{1}{9}\lvert a_3+2a_1\rvert^2$	$\lvert 1/\sqrt{6}\,b_0\cdot\tfrac{1}{2}b_1\rvert^2$	45	5.5	33	17
$\to K^0n\pi^-$	226	$nn\pi^0$	$K^0\pi^0 \to K^0\pi^-$	$\frac{2}{9}\lvert a_3-a_1\rvert^2$	$\frac{1}{2}\lvert b_1\rvert^2$	11	5.5	0	17
$\to K^0p\pi^0$	219	$np\pi^0$	$K^+\pi^- \to K^0\pi^0$	$\frac{2}{9}\lvert a_3-a_1\rvert^2$	$\frac{1}{2}\lvert b_1\rvert^2$	22	11	0	33

a) The corresponding processes with incident K^- can be obtained simply by the interchanges: $p\leftrightarrow n$; $K^+\leftrightarrow\bar K$; $\pi^+\leftrightarrow\pi^-$.
b) a_1, a_3 represent the scattering amplitudes relative to the two isospin eigenstates $t=\frac{1}{2}$, $t=\frac{3}{2}$ of the $K\pi$-system.
c) See Table II.

TABLE II (a).

Overall process	Kinetic energy threshold (MeV)	$\left(M_1 M_2/t_2\right)$ vertex	Peripheral reaction	Isospin analysis of the peripheral cross section (b)	Branching ratio (%) for the overall processes in the hypothesis of interaction lagrangian	
					(1)	(1')
$\pi^+p \to K^+\bar K^0 p$	1.356	$pp\pi^0$	$\pi^+\pi^0 \to K^+\bar K^0$	$\frac{1}{2}\lvert b_1\rvert^2$	0	25
$\pi^-p \to K^+K^-n$	1.358	$pn\pi^+$	$\pi^-\pi^+ \to K^+K^-$	$\frac{1}{6}\lvert b_0\rvert^2+\frac{1}{4}\lvert b_1\rvert^2$	50	25
$\to K^0\bar K^0 n$	1.358	$pn\pi^+$	$\pi^-\pi^+ \to K^0\bar K^0$	$\frac{1}{6}\lvert b_0\rvert^2+\frac{1}{4}\lvert b_1\rvert^2$	50	25
$\to K^0 K^- p$	1.356	$pp\pi^0$	$\pi^-\pi^0 \to K^0 K^-$	$\frac{1}{2}\lvert b_1\rvert^2$	0	25

a) The corresponding processes on neutrons can be obtained simply by the interchanges: $p\leftrightarrow n$; $\pi^+\leftrightarrow\pi^-$; $K\leftrightarrow\bar K$.
b) b_0, b_1 represent the reaction amplitudes relative to the two isospin eigenstates $t=0$ $t=1$ of the $K\bar K$-system.

different in the two physical processes $K + \pi \to K + \pi$ and $\pi + \pi \to K + \overline{K}$; in fact, they are different limits of the same analytic function.

From the isospin analysis of the peripheral reactions, we can obtain by means of (4) the branching ratios among the various overall processes of each table in the hypothesis that the K^2-π^2 coupling is described either by the Lagrangian (1) or (1'); or by assuming, from a phenomenological viewpoint, that the reaction $K + \pi \to K + \pi$ (Tables I and IV) occurs predominantly in one of the two isospin states $t = \frac{1}{2}$, $t = \frac{3}{2}$. We report the values of the branching ratios only for the processes listed in the Tables I and II. For the other processes (Tables III and IV) the branching ratios are functions of the momentum q_{1L} of the incident particle; one could evaluate them, for a fixed value of q_{1L}, by taking into account the kinematical factors occurring in (4) which now are different also for processes of the same kind.

TABLE III.

Overall process	Kinetic energy threshold (MeV)	$(M_1 M_2 \mu_2)$ vertex	Peripheral reaction	Isospin analysis of the peripheral cross section (a)
$K^- p \to \Lambda^0 \pi^+ \pi^-$	(exothermic)	$p \Lambda^0 K$	$K^- K \to \pi^+ \pi^-$	$\frac{1}{6}|b_0|^2 + \frac{1}{4}|b_1|^2$
$\to \Lambda^0 \pi^0 \pi^0$	(exothermic)	$p \Lambda^0 K^+$	$K^- K \to \pi^0 \pi^0$	$\frac{1}{6}|b_0|^2$
$\to \Sigma^0 \pi^+ \pi^-$	53.3	$p \Sigma^0 K^+$	$K^- K^+ \to \pi^+ \pi^-$	$\frac{1}{6}|b_0|^2 + \frac{1}{4}|b_1|^2$
$\to \Sigma^0 \pi^0 \pi^0$	38.9	$p \Sigma^0 K^+$	$K^- K^+ \to \pi^0 \pi^0$	$\frac{1}{6}|b_0|^2$
$\to \Sigma^- \pi^- \pi^0$	49.2	$p \Sigma^+ K^0$	$K^- K^0 \to \pi^0 \pi^-$	$|b_1|^2$
$\to \Sigma^- \pi^+ \pi^0$	61.7	—	—	—
$K^- n \to \Lambda^0 \pi^- \pi^0$	(exothermic)	$n \Lambda^0 K^0$	$K^- K^0 \to \pi^0 \pi^-$	$|b_1|^2$
$\to \Sigma^0 \pi^- \pi^0$	44.3	$n \Sigma^0 K^0$	$K^- K^0 \to \pi^0 \pi^-$	$|b_1|^2$
$\to \Sigma^- \pi^0 \pi^0$	52.8	$n \Sigma^- K$	$K^- K \to \pi^0 \pi^0$	$\frac{1}{6}|b_0|^2$
$\to \Sigma^- \pi^+ \pi^-$	67.2	$n \Sigma^- K^+$	$K^- K \to \pi^+ \pi^-$	$\frac{1}{6}|b_0|^2 + \frac{1}{4}|b_1|^2$
$\to \Sigma^+ \pi^- \pi^-$	54.6	—	—	—

a) See Table II.

We try now to get the order of magnitude of a « mean » cross-section for the peripheral reactions. This could be obtained by means of (4) from the knowledge of the experimental cross-section $\sigma(W)$ for the overall reactions. The scanty experimental information is restricted to some processes of type (I), (III) and (IV). For the processes listed in Table III the Chew and Low's conjecture can give however only some information at high energy well above

the threshold. This is due to the fact that the physical region for the peripheral process is defined by $w < 2\mu_K$ ([25]).

TABLE IV ([a]).

Overall process	Kinetic energy threshold (MeV)	$(M_1 M_2 \mu_2)$ vertex	Peripheral reaction	Isospin analysis of the peripheral cross section ([b])	Peripheralc cross section in terms of the b_0, b_1 amplitudes ([b])
$\pi^+ p \to K^+ \Sigma^+ \pi^0$	1.140	$p\Sigma^+ K^0$	$\pi^+ K^0 \to \pi^0 K^+$	$\frac{2}{9}\|a_3 - a_1\|^2$	$\frac{1}{2}\|b_1\|^2$
$\to K^0 \Sigma^+ \pi^+$	1.150	$p\Sigma^+ K^0$	$\pi^+ K^0 \to \pi^+ K^0$	$\frac{1}{9}\|a_3 + 2a_1\|^2$	$\left\|\frac{1}{\sqrt{6}}b_0 - \frac{1}{2}b_1\right\|^2$
$\to K^+ \Sigma^0 \pi^+$	1.146	$p\Sigma^0 K^+$	$\pi^+ K^+ \to \pi^+ K^+$	$\|a_3\|^2$	$\left\|\frac{1}{\sqrt{6}}b_0 - \frac{1}{2}b_1\right\|^2$
$\to K^+ \Lambda^0 \pi^+$	1.009	$p\Lambda^0 K^+$	$\pi^+ K^+ \to \pi^+ K^+$	$\|a_3\|^2$	$\left\|\frac{1}{\sqrt{6}}b_0 - \frac{1}{2}b_1\right\|^2$
$\pi^- p \to K^0 \Sigma^- \pi^-$	1.150	$p\Sigma^+ K^0$	$\pi^- K^0 \to \pi^- K^0$	$\|a_3\|^2$	$\left\|\sqrt{\frac{1}{6}}b_0 - \frac{1}{2}b_1\right\|^2$
$\to K^- \Sigma^0 \pi^-$	1.146	$p\Sigma^0 K^+$	$\pi^- K^+ \to \pi^- K^+$	$\frac{1}{9}\|a_3 + 2a_1\|^2$	$\left\|\frac{1}{\sqrt{6}}b_0 - \frac{1}{2}b_1\right\|^2$
$\to K^0 \Sigma^0 \pi^0$	1.137	$p\Sigma^0 K^+$	$\pi^- K^+ \to \pi^0 K^0$	$\frac{2}{9}\|a_3 - a_1\|^2$	$\frac{1}{2}\|b_1\|^2$
$\to K^0 \Lambda^0 \pi^0$	1.000	$p\Lambda^0 K^+$	$\pi^- K^+ \to \pi^0 K^0$	$\frac{2}{9}\|a_3 - a_1\|^2$	$\frac{1}{2}\|b_1\|^2$
$\to K^- \Lambda^0 \pi^-$	1.009	$p\Lambda^0 K^+$	$\pi^- K^+ \to \pi^- K^+$	$\frac{1}{9}\|a_3 + 2a_1\|^2$	$\left\|\frac{1}{\sqrt{6}}b_0 - \frac{1}{2}b_1\right\|^2$
$\to K^- \Sigma^- \pi^0$	1.156	—	—	—	—
$\to K^0 \Sigma^- \pi^+$	1.165	—	—	—	—

a) The corresponding processes on neutrons can be obtained simply by the interchanges: $p \leftrightarrow n$; $K^+ \leftrightarrow K^0$; $\Sigma^+ \leftrightarrow \Sigma^-$; $\pi^+ \leftrightarrow \pi^-$.

b) See Table I.

There are some indications that the total cross-section for reactions of kind (I) (specifically $K^+ + n \to K + \mathcal{N} + \pi^-$) ([26]), for a momentum of the incident K-meson equal to $q_{1L} \approx 600$ MeV/c (corresponding to $W \approx 1610$ MeV

([25]) The minimum of W compatible with the lower limit of w is $W_0 = 2\mu_K + \mu_Y$, while the value at threshold is $W_{th} = 2\mu_\pi + \mu_Y$.

([26]) M. GRILLI, L. GUERRIERO, M. MERLIN, Z. O'FRIEL and G. A. SALANDIN: Nuovo Cimento, 14, 704 (1959); B. SECHI-ZORN and G. T. ZORN: UNL preprint, to be published on the Phys. Rev. (1960).

and to $635 \leqslant w \leqslant 670$ MeV) should be of the order of $1/7$ millibarn. From this value we obtain by means of (4) $\bar{\sigma}_{K\pi \to K\pi} \approx 30$ millibarn.

Reactions of kind IV (in particular: $\pi^- + p \to \Lambda^0 + K^+ + \pi^-$) have been observed by BLAU et al. [27]; the experimental cross-section at $q_{1L} \approx 1290$ MeV/c is estimated to be of the order of 0.5 millibarn. If we make use of the previous result obtained for $\bar{\sigma}_{K\pi \to K\pi}$, we get from (6) in the present case ($W \approx 1830$ MeV; $630 \leqslant w \leqslant 730$ MeV): $\sigma \approx 3 \cdot 10^{-4}$ millibarn in the pseudoscalar case and $\sigma \approx 10^{-4}$ or 10^{-3} millibarn in the scalar case.

Reactions of kind (III) have been reported by ALVAREZ [28]: at $q_{1L} \approx 1.8$ GeV/c the total cross-section for the process $K^- + p \to \Lambda^0 + \pi^+ + \pi^+$ is of the order of a few millibarn. W is just above the lower limit W_0 for the applicability of (2) so that the use of (6) is now unreliable; a very rough estimate would give an extremely high value for the total cross-section of the peripheral process (c) at $w \approx 1$ GeV.

The results can be qualitatively expressed as follows: while the value of $\bar{\sigma}_{\mu\mu}$ obtained from the reactions of kind (I) might be reasonable, an extremely high value—which seems to be very unlikely—is required for $\bar{\sigma}_{\mu\mu}$ in order to reproduce by the described procedure the order of magnitude of the experimental cross-sections of processes (III) and (IV).

Notwithstanding the poor approximation used (e.g. the range of variability for w is different, and the cross-section $\sigma_{\mu\mu}$ is averaged in a different way in the various processes) and the scanty experimental information, it seems reasonable to say that it is not possible to explain consistently the reactions (I) (III) and (IV) by means of a direct K^2-π^2 interaction and the Chew and Low's conjectures. On one hand, the extrapolation of the relation (2) to the physical region might be no longer justifiable for the processes in which the intermediate particle is a K-meson (reactions (III) and (IV)), as previously pointed out. On the other hand, for some processes a « direct » interaction between the incident meson and the « core » of the baryon target might be important. In particular, the processes of kind (I) might be interpreted, in analogy with the pion production by pions [29], in terms of the excitation and decay of the $(\frac{3}{2}, \frac{3}{2})$ nucleon isobar [30]. An analysis of this type on a new set of experimental data on the reaction $K^+ + n \to K + N + \pi^\pm$ is now in progress [31]. In this connection, it will be very useful to compare the experimental branching ratios for the different reactions with those predicted by the $(\frac{3}{2}, \frac{3}{2})$

[27] M. BLAU, C. F. CARTER and A. PERLMUTTER: Nuovo Cimento, 14, 704 (1959).
[28] L. W. ALVAREZ: Proc. of the Annual Conference on High Energy Physics (Kiev, 1959).
[29] V. ALLES-BORELLI, S. BERGIA, E. PEREZ FERREIRA and P. WALOSCHEK: Nuovo Cimento, 14, 211 (1959).
[30] S. J. LINDENBAUM and R. B. STERNHEIMER: Phys. Rev., 105, 1874 (1957).
[31] W. F. FRY: private communication.

nucleon isobar model and by the present scheme. Evidence of the effect of the peripheral interactions should be searched in those processes in which the nucleon isobar effects are estimated to be relatively small.

At present, we conclude by saying that, by means of a direct K^2-π^2 coupling of comparable strength with that presumed for the π^4-coupling [9,10,18,20,21] it might be possible to reproduce—unlike the perturbation results—the experimental situation for the processes (I). Such a possibility seems, instead, to be ruled out for the reactions (III) and (IV). No experimental data are available, at present, for the reactions (II). However, if we remember that for such processes (as in the case of processes (I)) the intermediate particle is a pion, and that the application of the Chew and Low's conjecture has already given satisfactory results [18,20] in similar cases, we should expect that also reactions (II) occur in an extent in consistence with the proposed scheme.

Then, for such processes, we can try to go a little further. We can extract from the expression for the cross-section relative to the peripheral process (a) all those factors which must occur for kinematical reasons [32]. Thus we can write

$$(5) \qquad \sigma_{K\pi \to K\pi}(w) = \frac{1}{4w^2}\mathcal{M}_a(w),$$

where $\mathcal{M}_a$ is a dimensionless squared matrix element, dependent in general on w, it reduces simply to $\lambda^2/4\pi$ in a first order perturbation calculation based on the Lagrangian (1).

Similarly, for process (b) we obtain

$$(6) \qquad \sigma_{\pi\pi \to K\bar{K}}(w) = \frac{1}{4w^2}\left(\frac{w^2 - 4\mu_K^2}{w^2 - 4\mu_\pi^2}\right)^{\frac{1}{2}}\mathcal{M}_b(w).$$

We can now introduce a « mean » value $\bar{\mathcal{M}}(W)$ for $\mathcal{M}(w)$ in the same way as done for $\sigma_{\mu\mu}$; we can then replace (4) by

$$(7) \qquad \sigma(W) = \alpha\frac{j_\pi^2}{\mu_\pi^2 q_\pi^2}\,S(W)\bar{\mathcal{M}}(W).$$

We report in Fig. 4 the quantity $\sigma/\alpha\bar{\mathcal{M}}$ as a function of W for the processes (I) and (II). A comparison with experimental data—in the hypothesis of a dominant direct K^2-π^2 coupling—should give some information about the behaviour of $\mathcal{M}$. In this connection a comparison between the cross-sections σ_I

[32] V. F. WEISSKOPF: *Introduction to field theory*, in *Lectures given at CERN* (1958); B. D'ESPAGNAT, J. M. JAUCH and Y. YAMAGUCHI: *Lectures given at CERN on Strange Particles Physics* (1958-59).

and σ_{II} should be important: *e.g.* for kinetic energies ranging from zero to about 400 MeV above the threshold, with the assumption $\mathcal{M}_a \approx \mathcal{M}_b$, we obtain $\sigma_{\text{II}}/\sigma_{\text{I}} \approx 0.2$.
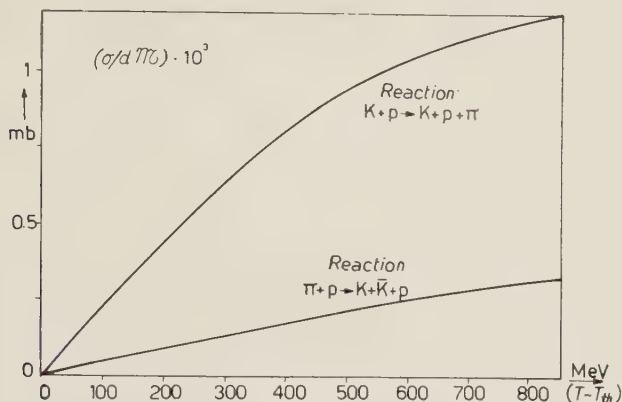

Fig. 4. – Ratios $\sigma/\alpha \mathcal{M}$ for reactions of types I and II, as functions of the kinetic energy above threshold $(T - T_{\text{th}})$.

If we refer to the above mentioned experimental indication relative to process (I), we see that a (first order) perturbation calculation of $\mathcal{M}(w)$ is meaningless: in fact, according to (5), a coupling constant $\lambda^2/4\pi \gg 1$ would be required in order to obtain a peripheral cross-section of the order of 30 mb. This appears also from the work by OKUBO [33], who obtained an effective range formula for the K-π scattering. The matrix element $\mathcal{M}(w)$ should be affected rather strongly by the π^4-coupling, and the presence of possible resonances [18,21] might be important for some processes. Investigations based on the double dispersion representation have already been carried out for the block KKππ [34] and its implications in the K-nucleon interactions have been examined [35]. In this connection, when experimental data will be available, the analysis of some of the reactions listed in the Tables I and II might provide very useful information.

[33] S. OKUBO: *Phys. Rev.*, **118**, 357 (1960).

[34] B. W. LEE: *Theory of the kaon-pion interaction*, preprint (1960); M. GOURDIN, Y. NOIROT and PH. SALIN: *Diffusion meson π-meson K*, preprint (1959).

[35] F. FERRARI, G. FRYE and M. PUSTERLA: *Effect of the pion-pion resonance on K-p scattering*, UCRL-9196 (1960); F. FERRARI, M. NAUENBERG and M. PUSTERLA: *Low energy dependence of π-Y and K-nucleon S-wave amplitude*, UCRL-8985 (1959); M. GOURDIN and M. RIMPAULT: *Production associée de particules etranges*, preprint (1959).

* * *

RIASSUNTO

Si studia l'effetto delle interazioni dirette fra pioni e mesoni K nei processi di diffusione anelastica π-N e K-N. Si suppone che il bosone incidente interagisca solo coi pioni periferici della nuvola del nucleone. La sezione d'urto totale si esprime in funzione della sezione d'urto per l'interazione diretta pione-mesone K, mediata sull'energia, secondo il procedimento di Chew e Low. Si discute l'attendibilità della approssimazione e si danno, pei vari casi trattati, i rapporti fra le probabilità delle diverse reazioni.

K⁻-Meson Capture by Helium (*).

T. B. Day

Physics Department, University of Maryland - College Park, Md.

Summary. — Some atomic and molecular processes which occur when K⁻-mesons stop and are captured in liquid helium are investigated. It is shown that while the K⁻-meson is in its initially highly-excited atomic states, S-state capture is predominant.

---

1. – Introduction.

Some understanding of the atomic and molecular processes which occur when K⁻-mesons stop in liquid bubble chambers is preliminary to any discussion of the direct nuclear capture which finally results. Not only do these atomic and molecular processes have intrinsic interest, but, in particular, a knowledge of the atomic orbital from which the meson is captured has a direct bearing on the interpretation of the nuclear capture processes.

Fairly recently, Day, Snow and Sucher considered the case of K⁻-mesons being captured in liquid hydrogen [1,2]. They showed that for the excited states of the (K⁻, p) atom through which the K⁻-meson must pass, the electric field of a neighboring proton will mix degenerate levels of the atom in such a way that the very strong S-state nuclear capture will completely dominate [3].

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[1] T. B. Day, G. A. Snow and J. Sucher: *Phys. Rev. Lett.*, **3**, 61 (1959).

[2] T. B. Day, G. A. Snow and J. Sucher: *Phys. Rev.*, **118**, 864 (1960). In particular, footnote [11].

[3] The arguments in references [1] and [2] were semi-classical. A quantum mechanical calculation of the effect for a very closely related model has been performed by F. Gürsey and C. N. Yang with results which agree with references [1] and [2]. F. Gürsey: private communication.

Experimental difficulties have, as yet, prevented the direct verification of this effect (by observing the lack of K_α-mesic X-rays, for example). Still, the theoretical arguments are sufficiently convincing that one can, with fair confidence, assume that the interaction of K⁻-mesons with the protons of liquid hydrogen occurs only in S-states. This has many important consequences [1], and in particular, has been used recently in a measurement of the spin of the Σ^+ hyperon [4].

In the case of liquid helium bubble chambers, the problem is completely different. For, the simplicity of the considerations for hydrogen relied on the fact that the (K⁻, p) atom was a « small, neutral object » which could wander within another hydrogen atom's electronic orbit and thus feel the intense electric field of the proton. The object which results when a K⁻-meson is stopped by a helium atom very quickly becomes a small, charged ion. Thus it will not experience the simple Stark effect considered for hydrogen.

This, then, leads to a consideration of the various atomic and molecular processes which the resulting ion undergoes in liquid helium, and which are dealt with in turn in the succeeding sections. Finally, we conclude by pointing out that, at least in the initial stages of the de-excitation of the K⁻-meson the most likely process leads again to S-state capture, and we discuss the usefulness of this fact in establishing the relative K⁻-Λ^0 parity.

2. – Initial atomic capture and Auger.

When the K⁻-meson is first captured by a helium atom, it tends to occupy an excited state whose wave function best overlaps that of the electron which is replaced [5]. Since the reduced mass μ_k of the K⁻-meson in a helium atom is 856 m_e, the principal quantum number, n, of the initially occupied state is

$$(1) \qquad\qquad n_1 = \sqrt{856} \sim 30 \ .$$

In such a neutral atom, where the meson very closely overlaps the second electron, the Auger process is highly likely, and occurs in times of the order 10^{-15} s [6]. The principal quantum number of the state about the unshielded α-particle to which the K⁻-meson must fall in order to release the 24.56 eV

[4] J. LEITNER, P. NORDIN jr., A. H. ROSENFELD, F. T. SOLMITZ and R. D. TRIPP: Phys. Rev. Lett., 3, 238 (1959).

[5] G. A. BAKER jr.: Phys. Rev., 117, 1130 (1960).

[6] E. H. S. BURHOP: The Auger Effect and Other Radiationless Transitions (Cambridge, 1952), chap. VII.

electron ionization energy [7] is

$$(2) \qquad\qquad n_2 \sim 25 \, .$$

In such a state, for circular orbits, the average value of the radius is

$$(3) \qquad\qquad \langle r_2 \rangle = \frac{n_2^2}{Z} \frac{m_e}{\mu_k} a_0 \; = 0.3 \, a_0 \, ,$$

(where a_0 is the electronic Bohr radius) which should be compared with the average radius of an electron in a helium atom (using the best hydrogen wave functions, with $Z_{\text{eff}} = 1.69$ [8])

$$(4) \qquad\qquad \langle r_1 \rangle = \frac{n_1^2}{Z_{\text{eff}}} \frac{m_e}{\mu_k} a_0 = 0.6 \, a_0 \, .$$

For such a $(K^-, \alpha)^+$ ion, which looks roughly like a heavy proton to the surrounding liquid, it is energetically impossible to pick up another electron from a neighboring helium atom [9,10]. Thus, we must consider processes in which this $(K^-, \alpha)^+$ ion participates as a relatively small, charged object.

3. – Radiation and nuclear capture.

For an isolated, highly excited, $(K^-, \alpha)^+$ ion, two processes are possible: de-excitation with emission of radiation, and nuclear capture. The first of these, radiation, is easily estimated from the tables in Bethe and Salpeter [11], using the value of the meson reduced mass $\mu_k = 856 \, m_e$, and the charge $Z = 2$. We get, for the fastest radiative transitions, the $nP \to 1S$, for $n \geqslant 6$

$$(5) \qquad\qquad \Gamma_{\text{rad}}(nP) \sim \frac{6 \cdot 10^{13}}{n^3} \; \text{s}^{-1} \, .$$

For an atom in a pure S-state, the nuclear capture rate is estimated directly from a knowledge of the rate in hydrogen [12], by making the assumption of

[7] K. R. ATKINS: *Liquid Helium* (Cambridge, 1959).

[8] L. I. SCHIFF: *Quantum Mechanics*, 2nd ed. (New York, 1955), p. 176.

[9] S. K. ALLISON: *Rev. Mod. Phys.*, **30**, 1137 (1958).

[10] H. S. W. MASSEY and E. H. S. BURHOP: *Electronic and Ionic Impact Phenomena* (Oxford, 1952).

[11] H. A. BETHE and E. E. SALPETER: *Quantum Mechanics of One- and Two-Electron Atoms* (Berlin, 1957), Table 15.

[12] R. H. DALITZ and S. F. TUAN: *Ann. Phys.*, **6**, no. 3 (1959); and B. R. DESAI: unpublished Lawrence Radiation Laboratory report UCRL-9014, University of California, footnote [12].

roughly equal interaction of the K^--meson with protons and neutrons. We then get

(6)
$$\Gamma_{cap}(nS) \sim \frac{2 \cdot 10^{19}}{n^3} \; s^{-1} \, .$$

Since the rate of direct capture due to S-wave interaction is crudely proportional to the integral of the square of the wave function of the meson within the nuclear volume, a direct calculation, using hydrogenic wave functions, gives

(7)
$$\frac{\Gamma_{cap}(nP)}{\Gamma_{cap}(nS)} = \frac{1}{15}\left(\frac{ZR_\alpha}{a_0}\frac{\mu_{\hbar}}{m_e}\right)^2\left(1 - \frac{1}{n^2}\right),$$

or, using equation (6) and a reasonable radius R_α for the α-particle,

(8)
$$\Gamma_{cap}(nP) \sim \frac{4 \cdot 10^{15}}{n^3} s^{-1} \, .$$

This, then, is the rate of direct capture of a K^--meson from an (nP) state about an α-particle, due to S-wave interaction with the nucleons (*i.e.* due to the finite size of the nucleus). Similar calculations would give the rates for S-wave interaction capture from D and higher levels.

4. – Polarization capture and external Auger.

We next consider processes which necessitate the presence of another helium atom. For the moment, we consider the $(K^-, \alpha)^+$ ion at a fixed distance, R, from the nearby helium atom. Two of these processes are closely related to each other, and to the usual calculation of Van der Waal's forces between atoms ([13]). (See reference (⁵), p. 177.) We will call them, for convenience the polarization capture, and the external Auger processes, respectively. Actually, of course, they, as well as the Van der Waal's force, are all just a manifestation of the Coulomb interaction between the various charged particles.

4'1. *Polarization capture.* – This process is one in which a meson in the nP state is captured from the nS state by first polarizing a nearby helium atom ([14]). Thus, for a $(K^-, \alpha)^+$ ion in an nP state interacting with a nearby helium atom at a fixed distance, R, there is a certain probability of finding

([13]) T. B. DAY: *Bull. Am. Phys. Soc.*, **5**, 225 (1960) and University of Maryland Physics Department Technical Report no. 175.

([14]) This process was considered in another connection by M. A. RUDERMAN: *Phys. Rev.*, **118**, 1632 (1960).

the ion in an nS state. This probability is given simply by

$$(9) \qquad f = \text{fraction in } nS = \sum_N \frac{|\langle N|H'|0\rangle|^2}{(E_N - E_0)^2} .$$

Here, $\langle N|$ is a state with the electrons in the nearby helium atom excited, and the meson in its nS state; $|0\rangle$ is a state with the helium atom in its ground state, and the meson in the nP state. H' is the interaction between the meson and the electrons and, for large ion-atom separation, may be expanded as

$$(10) \qquad H' = \frac{e^2}{R^3}\left[(\boldsymbol{r}_{e_1} + \boldsymbol{r}_{e_2})\cdot\boldsymbol{r}_\mu - 3(\boldsymbol{r}_{e_1} + \boldsymbol{r}_{e_2})\cdot\boldsymbol{R}\boldsymbol{r}_\mu\cdot\boldsymbol{R}\right] .$$

The subscripts on the coordinates refer to electron one and two, and K⁻-meson, respectively. The first term in the bracket gives rise to virtual transitions wherein the angular momentum quantum number 1 changes by ± 1, and the magnetic quantum number m_1 does not change. The second term gives rise to virtual transitions with $\Delta l = \pm 1$, $\Delta m_1 = \pm 1$. In order to compute the nuclear capture from the nP state *via* the nS state by this mechanism, one simply multiplies the fraction eq. (9) by the nS capture rate, eq. (6). (The goodness of this approximation is discussed by Ruderman.[14]).

To get an estimate for the fraction of eq. (9) we first note that the perturbation is additive in the electron coordinates, so there are no states $\langle N|$ where both electrons are excited. Thus, we need perform the sum only for one electron, and multiply by two. Next, in performing the sum over m_1, we get simply a sum of terms with matrix elements like $|\langle N|Z_e Z_\mu|0\rangle|^2$. Taking account of the numerical factors, we get

$$(11) \qquad f = \frac{e^4}{R^6} 12 \sum_N \frac{|\langle N|Z_e Z_\mu|0\rangle|^2}{(E_N - E_0)^2} = 12\frac{e^4}{R^6}|\langle nS|Z_\mu|nP\rangle|^2 \sum_N \frac{|\langle N_e|Z_e|0_e\rangle|^2}{(E_N - E_0)^2} ,$$

since the matrix element factors. Using reference [11], p. 258 ff, and getting an upper and lower limit for f by putting in the minimum and average value of $(E_N - E_0)$ respectively, we find (assuming the electron to be in a hydrogen-like atom of charge Z_{eff})

$$(12) \qquad \frac{27}{(R/a_0)^6}\frac{m_e^2}{\mu_k^2 Z^2 Z_{eff}^6} < f < \frac{48}{(R/a_0)^6}\frac{m_e^2}{\mu_k^2 Z^2 Z_{eff}^6} .$$

Combining this with eq. (6), we find a capture rate from the nP state *via* the S-state by this polarization mechanism of

$$(13) \qquad \Gamma_{p1}(nS) < \frac{9\cdot 10^{12}}{(R/a_0)^6}\left(n - \frac{1}{n}\right)\text{s}^{-1} .$$

4'2. *External Auger.* – The external Auger process, in which the meson de-excites to a lower state, ejecting an electron of a nearby helium atom into the continuum, is closely related to both the Van der Waal's calculation and the above polarization capture. Again considering the case of fixed ion-atom distance R, the transition rate for the meson to go from state n, l to state n', l' while the electron goes from the $1S$ state to the continuum is just

$$(14) \qquad \Gamma_{\text{Aug}}(n,\, l \to n',\, l') = \frac{2\pi}{\hbar}\, \varrho_f \,|\langle n',\, l';\, \text{cont.}\,|H'|\,n,\, l;\, 1S\rangle|^2\,.$$

Here, ϱ_f is the density of final states for the electron and H' is given by eq. (10). The calculation proceeds exactly as for the polarization capture immediately above. Using plane waves for the electron continuum wave-function in the electron-coordinate part of the matrix element, we get

$$(15) \qquad \Gamma_{\text{Aug}}(n,\, l \to n',\, l-1) = \frac{1.7\cdot 10^{12}}{(R/a_0)^6}\, \frac{K^3}{(1+K^2)^6}\left(\frac{l}{2l+1}\right)(R_{nl}^{n'l-1})^2\ \text{s}^{-1}\,,$$

where $R_{nl}^{n',l-1}$ is the radial dipole matrix element in ordinary atomic units (see reference [11], eq. (60.8));

$$(16) \qquad\qquad K = \frac{ka_0}{Z_{\text{eff}}} = 0.16\,(E_{\text{f.n}}(\text{volt}))^{\frac{1}{2}}\,,$$

where k is the wave number of the electron in the continuum, and $E_{\text{fin}}(\text{volts})$ its energy expressed in electron volts. In eq. (15) (and in what follows) we consider only transitions with $\Delta l = -1$ as being the most likely (see reference [11], Table XIII), and the appropriate sum and average over, respectively, final and initial magnetic quantum numbers have been performed.

The external Auger rates, eq. (15), fall off very rapidly with large changes in principal quantum number n. To get a crude estimate of the rates for eq. (15), we take a statistical average of the rates for the initial level, and, when possible, use the tabulated values of $R_{nl}^{n'l-1}$ given in reference [11], Table XIII. (This table was extended to $n = 7$ using the table of Maxwell [15].) Defining the statistical average over the initial level by

$$(17) \qquad\qquad \sum_{l=0}^{n-1} \frac{2l+1}{n^2}\, \Gamma_{\text{Aug}}(n,\, l \to n',\, l-1) \equiv \Gamma_{\text{Aug}}(\langle n\rangle)\,,$$

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[15] L. R. MAXWELL: *Phys. Rev.*, **38**, 1664 (1931).

we get

$$(18) \quad \begin{cases} \Gamma_{\text{Aug}}(\langle 5 \rangle) = \dfrac{4.4 \cdot 10^7}{(R/a_0)^6} \ \text{s}^{-1}, \\[12pt] \Gamma_{\text{Aug}}(\langle 6 \rangle) = \dfrac{1.4 \cdot 10^9}{(R/a_0)^6} \ \text{s}^{-1}, \\[12pt] \Gamma_{\text{Aug}}(\langle 7 \rangle) = \dfrac{2.5 \cdot 10^{10}}{(R/a_0)^6} \ \text{s}^{-1}. \end{cases}$$

For higher values of n, where the dipole matrix elements are not tabulated and where very many of them are involved, we use the approximation (for transitions where $\Delta n = -1$)

$$(19) \quad \sum_l \left(\frac{2l+1}{n^2} \right) \left[\frac{l}{2l+1} (R_{n,l}^{n-1,l-1})^2 \right] \to \frac{n^4}{6.5}.$$

This is the dependence on n which one expects from the fact that for large n, $R_{n,l}^{n-1,l-1}$ goes as n^2. It is also roughly in agreement with the n-dependence of the average lifetimes of highly excited states (reference [11], p. 269). The value of the coefficient of n^4 was found by plotting the left-hand side of eq. (19) for the tabulated values of n. For still higher values of n, where changes in n must be greater than 1 in order to release enough energy, we reduce the estimate, eq. (19), by a factor of five to ten depending on the size of the jump in Δn. (Compare with Table XIII, reference [11], for $\Delta n > 1$.) Some typical results are

$$(20) \quad \begin{cases} \Gamma_{\text{Aug}}(\langle 10 \rangle) \sim \dfrac{5 \cdot 10^{12}}{(R/a_0)^6} \ \text{s}^{-1}, \\[12pt] \Gamma_{\text{Aug}}(\langle 20 \rangle) \sim \dfrac{5 \cdot 10^{13}}{(R/a_0)^6} \ \text{s}^{-1}, \\[12pt] \Gamma_{\text{Aug}}(\langle 25 \rangle) \sim \dfrac{5 \cdot 10^{14}}{(R/a_0)^6} \ \text{s}^{-1}. \end{cases}$$

5. – Molecular-field Stark effect.

There is a further possibility for a process which involves the $(K^-, \alpha)^+$ ion and a neighboring helium atom. It is possible that a meta-stable molecule might be formed. The $(K^-, \alpha)^+$ ion acts roughly like a heavy proton. The $(p, He)^+$ molecule is well-known [16,17], although the three-body recombination

problem involved in its formation has not been studied extensively in liquids ([18]). However, while these recombination times are expected to be very short, it is not essential for our purposes, here, that the molecule actually form. Rather, we make only the simpler assumption that the $(K^-, \alpha)^+$ ion feels the molecular potential while in collision with a nearby helium atom.

This potential between a helium atom and a proton is given in reference ([17]), Fig. 1, and represents an equilibrium separation of $1.44\,a_0$, and a potential depth at minimum of 0.14 Ry. The curve is well represented near the minimum by eq. (6) of reference ([17])

$$(21) \qquad \Phi(R) = 0.14 \left[\left(\frac{1.44\,a_0}{R} \right)^4 - 2 \left(\frac{1.44\,a_0}{R} \right)^2 \right] \text{Ry} .$$

By using the Hellman-Feynman theorem ([19]) we can differentiate this curve to get the electric field at the $(K^-, \alpha)^-$ ion due to the helium atom ([20]), and we have recovered our Stark effect.

The same calculation as in reference ([1]) may again be used, except that the electric field is now the molecular one given by the derivative of eq. (21). It should also be noted that, for the case of a $(K^-, \alpha)^+$ ion in an nP state, the Stark effect capture *via* S-state absorption is given by case (b)in footnote 9 of reference ([1]). This is due to the fact that the S-state capture is so much larger in our case, and also that the electric field is so much less. Thus, the Stark oscillation rate, ω_s, is again given by

$$(22) \qquad \omega_s \sim \langle nP \,|\, Z_\mu \,|\, nS \rangle \, \frac{\mathrm{d}\Phi(R)}{\mathrm{d}R} ,$$

where $\Phi(R)$ is given in eq. (21), and the dipole matrix element is the same as that used in the calculation in Sect. 4, above, and is

$$(23) \qquad \langle nP \,|\, Z_\mu \,|\, nS \rangle = \frac{\sqrt{3}}{2}\, n\sqrt{n^2 - 1} ,$$

in mesic-helium atomic units, *i.e.*

$$(24) \qquad \langle nP \,|\, Z_\mu \,|\, nS \rangle = 5 \cdot 10^{-4} n^2 \, \sqrt{1 - 1/n^2}\, a_0 .$$

Since, for almost all values of n considered here, the Stark oscillation rate is much less than the nS-state capture rate $\Gamma_{\text{cap}}(nS)$ of eq. (6), the Stark effect capture rate $\gamma_{\text{Stark}}(nS$ capture) for a $(K^-, \alpha)^-$ ion in a pure nP state *via* S-state

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([18]) See reference ([10]), chap. x.

([19]) H. HELLMAN: *Einführung in die Quantenchemie* (Leipzig, 1937), p. 285; R. P. FEYNMAN: *Phys. Rev.*, **56**, 340 (1939).

([20]) L. PAULING: *The Nature of the Chemical Bond*, 3rd ed. (Ithaca, 1960).

capture is [1]

$$(25) \qquad \gamma_{\text{Stark}}(nS \text{ capture}) = \frac{1}{4} \Gamma_{\text{cap}}(nS) \left| \frac{4\omega_S}{\Gamma_{\text{cap}}(nS)} \right|^2 .$$

Thus, for the molecular-field Stark effect capture *via* S-states, we get

$$(26) \qquad \gamma_{\text{Stark}}(nS \text{ capture}) \sim 3 \cdot 10^6 n^7 \left[\left(\frac{1.44 \, a_0}{R} \right)^3 - \left(\frac{1.44 \, a_0}{R} \right)^5 \right]^2 \text{s}^{-1} .$$

On the other hand, ignoring for a moment, the possibility of capture from the S-state, exactly the same process can lead to a mixing of levels with the P-state, from which there could also be direct nuclear capture. This direct capture rate from the nP state, eq. (8), is also large, but is still less than the Stark oscillation rate, eq. (22). Thus, case (*a*) of footnote 9 in reference [1] will apply, and the Stark effect capture from the D-states via P-state capture will be

$$(27) \qquad \gamma_{\text{Stark}}(nP \text{ capture}) \sim \frac{2 \cdot 10^{15}}{n^3} \text{ s}^{-1} .$$

Note, in particular, the different dependence on n, to which we will return in the concluding Section.

Finally, a factor of $1/n$ should be incorporated in these rates, since there are n degenerate levels with the magnetic quantum number $m_l = 0$ (for the S-state capture case) which are mixed and decay together in the static external electric field, when all states are populated statistically [2]. Thus, the capture rate *via* the nS-state for the $m_l = 0$ levels due to the Stark effect is

$$(28) \qquad \Gamma_{\text{Stark}}(nS \text{ capture}, m_l = 0) \sim 3 \cdot 10^6 n^6 \left[\left(\frac{1.44 \, a_0}{R} \right)^3 - \left(\frac{1.44 \, a_0}{R} \right)^5 \right]^2 \text{s}^{-1} .$$

A similar factor $1/n$ should multiply eq. (27) for the P-state Stark effect capture, (as well as eq. (13) for the polarization capture rate) [21].

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[21] Considerations similar to those of reference [1] may be applied to liquid deuterium. Here, however, direct P-state capture via S-wave interaction can occur as well. Using the notation of reference [2], the rates for liquid deuterium are

$$(a) \qquad \Gamma_{\text{rad}}(6P \to 1S) = 1.6 \cdot 10^{10} \text{ s}^{-1} ,$$

$$(b) \qquad \Gamma_{\text{coll}}^n(S\text{-Stark}) \sim \frac{3 \cdot 10^{12}}{n} \left(1 - \exp \left[-7.5 \cdot 10^3/n^4 \right] \right) ,$$

$$(c) \qquad \Gamma_{\text{coll}}^n(P\text{-Stark}) \sim \frac{3 \cdot 10^{12}}{n} \left(1 - \exp \left[-10/n^4 \right] \right) .$$

Thus, for $n \sim 10$,

$$\Gamma_{\text{rad}}(10P) \sim 10 \Gamma_{\text{coll}}^{10}(P\text{-Stark}) , \qquad \Gamma_{\text{coll}}^{10}(S\text{-Stark}) \sim 100 \Gamma_{\text{rad}}(10P) ,$$

and there will be no appreciable direct capture from P-states due to Stark mixing in deuterium.

6. – In flight averages.

In this Section, we drop the assumption that the $(K^-, \alpha)^+$ ion-helium atom distance is fixed. We consider the ion as moving slowly and uniformly by the atom at an impact parameter b. Then, at any distance R from the atom, the time-rate of change of the number N of ions in a given state due to some transition rate is given by

$$(29) \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = -\, \Gamma(R[t])N \, .$$

Thus, after the ion has gone by the atom, the final number N of ions left in the state which initially had N_0 ions is given by

$$(30) \qquad -\ln\left(\frac{N}{N_0}(b)\right) = 2\int\limits_{\substack{R(b)\\ \mathrm{min}}}^{R_{\mathrm{max}}} \Gamma(R[t]) \, \mathrm{d}t \, .$$

This expression is a function of the impact parameter, b, and is thus averaged over b by

$$(31) \qquad \left\langle -\ln\frac{N}{N_0} \right\rangle = \frac{1}{\pi R_{\mathrm{max}}^2} \, 2\pi \int\limits_0^{R_{\mathrm{max}}} b \, \mathrm{d}b \left[-\ln\left(\frac{N}{N_0}(b)\right) \right],$$

and set, by definition, equal to

$$(32) \qquad \left\langle -\ln\frac{N}{N_0} \right\rangle \equiv \langle \Gamma \rangle \langle t \rangle \, ,$$

where the average time of passage with uniform velocity v through a sphere of radius R_{max} is

$$(33) \qquad \langle t \rangle = \frac{2}{3}\left(\frac{2R_{\mathrm{max}}}{v}\right).$$

Eq. (31) defines an average transition rate $\langle \Gamma \rangle$, which we will then consider.

In performing the integrations of eq. (30) and (31), account must be taken of the fact that the ion-atom distance will not get less than some distance of closest approach, b_{min}: We choose this distance

$$(34) \qquad b_{\mathrm{min}} = 1 \, a_0 \, ,$$

as that ion-atom separation where the molecular potential turns positive (see Fig. 1, reference ([17])). This assumes that the ion has very little kinetic energy compared to the depth of the molecular well. We choose the maximum separation

$$(35) \qquad R_{max} = 3.6 \, a_0 \, ,$$

as the average mid-point between two helium atoms in liquid helium ([7]). It should be noted, from eqs. (30) and (33), that the average transition rate $\langle \Gamma \rangle$ defined with these approximations does not depend on the velocity (i.e. an appropriate cross-section would go as $1/v$).

Then, we can summarize the results of the integration in eqs. (30) and (31) for various instantaneous rates $\Gamma(R)$ by

$$(36) \qquad \begin{cases} \Gamma(R) = A/(R/a_0)^6 \,, & \langle \Gamma \rangle = 2.1 \cdot 10^{-2} A \text{ s}^{-1} \\ \Gamma(R) = B/(R/a_0)^8 \,, & \langle \Gamma \rangle = 1.3 \cdot 10^{-2} B \text{ s}^{-1} \\ \Gamma(R) = C/(R/a_0)^{10} \,, & \langle \Gamma \rangle = 0.92 \cdot 10^{-2} C \text{ s}^{-1} \,. \end{cases}$$

7. – Discussion.

By using eq. (36), we can define average transition rates for the various processes considered in Sect. 4 and 5. Thus, for the polarization capture process for a $(K^-, \alpha)^+$ ion in a pure nP state via the S-state,

$$(37) \qquad \langle \Gamma_{pol}(nS) \rangle \leqslant 2 \cdot 10^{11}(n - 1/n) \text{ s}^{-1} \,.$$

For the external Auger process, assuming statistical distribution and considering only those transitions most favored by the selection rules ($\Delta l - -1$) and allowed by energy conservation, we get

$$(38) \qquad \begin{cases} \langle \Gamma_{Aug}(\langle 5 \rangle) \rangle = 9 \cdot 10^5 \text{ s}^{-1} \\ \langle \Gamma_{Aug}(\langle 6 \rangle) \rangle = 3 \cdot 10^7 \text{ s}^{-1} \\ \langle \Gamma_{Aug}(\langle 7 \rangle) \rangle = 5 \cdot 10^8 \text{ s}^{-1} \\ \langle \Gamma_{Aug}(\langle 10 \rangle) \rangle \sim 1 \cdot 10^{11} \text{ s}^{-1} \\ \langle \Gamma_{Aug}(\langle 20 \rangle) \rangle \sim 1 \cdot 10^{12} \text{ s}^{-1} \\ \langle \Gamma_{Aug}(\langle 25 \rangle) \rangle \sim 5 \cdot 10^{12} \text{ s}^{-1} \,. \end{cases}$$

Finally, the molecular-field Stark effect capture via S-state capture gives

(for $m_l = 0$)

(39) $\langle \Gamma_{\text{Stark}}(nS \text{ capture}, m_l = 0)\rangle \sim 2 \cdot 10^5 n^6 \text{ s}^{-1}$,

while that *via* P-state capture did not depend on R, so

(40) $\Gamma_{\text{Stark}}(nP \text{ capture}, m_l = 0) \sim \dfrac{2 \cdot 10^{15}}{n^4} \text{ s}^{-1}$.

(The polarization capture process gives a roughly constant additional contribution to the S-state capture of eq. (39). The Stark effect, for $m_l = 0$, can mix levels with the nP state, from which polarization capture *via* the S-state can occur. This mixing requires an additional factor of $(\frac{1}{2}(1/n))$ to multiply eq. (37) in order to get the contribution to capture *via* S-states due to this process.)

In general, the Stark capture rates of eqs. (39) and (40) for $m_l = 0$ would have to be modified by some further factor which depended on the distribution of mesons in the levels of the atom, and on the directional properties of the electric field felt by the $(K^-, \alpha)^+$ ion. Such a factor might be as high as $(1/n)$ [2]. However, two circumstances would seem to argue that the factor of $(1/n)$, already necessarily incorporated in eqs. (39) and (40) because of the number of degenerate states involved in the Stark capture, is sufficient. The first is the fact that in liquid helium, positive ions lead to large clusters due to polarization forces [22]. This gives rise to weak electric fields whose directions change rapidly, which then destroys the $\Delta m_l = 0$ selection rule for the ordinary Stark mixing. The second circumstance arises in the initial capture of the K^--meson by helium atom. For, when this occurs, angular momentum barriers seem to prevent anything like a statistical distribution of mesons for the high l-states, but rather, favor the population of low and intermediate angular momentum sublevels (see, in particular, Fig. 2 of reference [5]). This kind of population will be accentuated by Auger transitions [23]. This has the effect not only of changing the factor $(1/n)$ incorporated in eqs. (39) and (40) to something like $(2/n)$, but also of decreasing the estimates of the Auger rates, which were given in eq. (38) for a statistical distribution [24].

If we restrict ourselves to the very high initial values of n, $n \sim 20 \div 30$, then clearly, by comparing the rates given by eq. (5) with those given by eqs. (37)-(40), radiation does not play any significant role. This leaves three

[22] G. Careri, U. Fasoli and F. S. Gaeta: *Nuovo Cimento*, **15**, 774 (1960); K. R. Atkins: *Phys. Rev.*, **116**, 1339 (1959); L. Meyer and F. Reif: *Phys. Rev. Lett.*, **5**, 1 (1960).

[23] M. Demeur: *Nucl. Phys.*, **1**, 516 (1956).

[24] The dipole matrix elements involved in the external Auger rates increase with increasing l for fixed initial and final principal quantum numbers. See ref. [11], p. 264.

competing channels to consider: the external Auger process, which de-excites
the ion to lower n-values without necessarily putting the meson in an S-state;
the direct capture from S-states *via* the mixing of levels due to the molecular-
field Stark effect (as well as due to a mixing into the nP state followed by a
polarization capture process); the direct capture from P-states *via* the mixing
of levels due to the molecular-field Stark effect. By comparing the rates of
eqs. (37)-(40), and by making some allowance for the clustering and initial
meson distribution discussed in the paragraph immediately above, we see
that for $n \sim 20 \div 30$, the S-state capture *via* the molecular-field Stark effect
will be $\sim 10 \div 50$ times faster than the nearst competing process, the external
Auger process. However, for lower n-values, *i.e.* $n \sim 10$, all the processes
(except radiation) become comparable, and the capture from the nP-state
via the Stark effect begins to become dominant. This is due to the different
behaviour with n of the Stark capture from nS- and nP-states. It is in contrast
with the hydrogen case, where there is no nP-state capture *via* S-wave inter-
action with the nucleon, only capture *via* the much smaller P-wave interaction;
and in contrast to the deuterium case of footnote ([21]), where the behaviour
with n of the Stark capture from nS- and nP-states is the same, and the
rate for nP capture is always smaller.

8. – Conclusions.

We see, then, that while the result is not as conclusive or convincing as
for the case of hydrogen studied in reference ([1]), it does indicate that when
K⁻-mesons stop in liquid helium, S-state capture will predominate. Just as
for hydrogen, this result could be checked experimentally by an absence of
mesic X-rays from the stopping K⁻-mesons.

Direct use can be made of this S-state capture in a way similar to that
for determining the Σ^+ spin from K⁻-mesons stopping in hydrogen ([4]). For,
when K⁻-mesons are captured by He, they can form the hyperfragments
$^4\text{He}_\Lambda$ and $^4\text{H}_\Lambda$. These in turn can undergo a two-body mesic decay.
By knowing that the K⁻-meson was captured from an S-state, a study of the
angular distributions of these decays can determine the spin of the hyper-
fragment. Then, if the spin of the $^4\text{He}_\Lambda$ or $^4\text{H}_\Lambda$ turns out to be zero, the
experiment of Block, *et al.* ([25]) determines directly that the relative K⁻-Λ^0
parity is odd.

([25]) M. M. Block E. B. Brucker, I. S. Hughes, T. Kikuchi, C. Meltze, F.
Anderson, A. Pevsner, E. M. Harth, J. Leitner and H. O. Cohh: *Phys. Rev.
Lett.*, **3**, 291 (1959). We are indebted to Professor M. M. Block for repeatedly empha-
sizing to us the importance of the S-state capture problem in helium.

* * *

We would like to thank Professor L. S. RODBERG and Professor J. SUCHER for several interesting discussions, and particularly Professor G. A. SNOW for his continuing and helpful interest.

RIASSUNTO (*)

Si esaminano alcuni processi atomici e molecolari che avvengono quando i mesoni K⁻ si fermano e vengono catturati nell'elio liquido. Si mostra che quando il mesone K è nei suoi stati atomici di alta eccitazione iniziale, la cattura nello stato S è predominante.

(*) *Traduzione a cura della Redazione.*

Radiative Corrections to Pair Production at High Energies.

K. MORK and H. OLSEN

Fysisk Institutt, Norges Tekniske Högskole - Trondheim

Recent measurements on the absorption of high energy photons in various elements [1] seem to indicate a deviation from the theory of Davies, Bethe and Maximon [2] of the order of one or two percents. It therefore becomes necessary to include into the theoretical considerations also the radiative corrections to the pair production process to the order e^2.

Previously expressions for the radiative corrections to the differential cross sections of pair production and bremsstrahlung have been derived by Fomin [3], by Mitra *et al.* [4] and by Mork [5]. The formulae are however so complicated even in the high energy approximation that the angular integrations become extremely difficult.

In order to avoid some of these complications we have used the method of virtual quanta [6] to calculate the radiative corrections to the spectral distribution of pair production and bremsstrahlung and to the total pair production cross section. It is simplest to consider the bremsstrahlung process and convert the results to pair production using the well known substitution law. There are two ways in which the radiative corrections enter, namely in the final state scattering of the emitted photon and as an increase of the number of virtual quanta. The first effect is taken into account by introducing into the semi-classical radiation formula the cross section for the radiative correction to the Compton effect given by BROWN and FEYNMAN [7] in the same manner as WILLIAMS and WEIZSÄCKER [6] originally introduced the KLEIN-NISHINA cross section.

The increase of the number of virtual quanta is due to the vacuum polarization of the nucleus and gives a change in the form factor of the Bethe

[1] E. MALAMUD: *Phys. Rev.*, **105**, 687 (1959), who also gives references to previous experiments.

[2] H. DAVIES, H. A. BETHE and L. C. MAXIMON: *Phys. Rev.*, **93**, 788 (1954).

[3] P. I. FOMIN: *Žurn. Exp. Teor. Fiz.*, **35**, 707 (1958) translation *Soviet Phys.*, **35**, 491 (1959).

[4] A. N. MITRA, P. NARAYANASWAMY and L. K. PANDE: *Nucl. Phys.*, **10**, 629 (1959).

[5] K. MORK: *Diploma work, Trondheim* (unpublished).

[6] E. J. WILLIAMS: *Kgl. Danske Videnskab. Selskab*, **13**, 4 (1935); C. F. v. WEIZSÄCKER: *Zeits. Phys.*, **88**, 612 (1934).

[7] L. M. BROWN and R. P. FEYNMAN: *Phys. Rev.*, **85**, 231 (1952).

Heitler cross section [8]. This part of the cross section has been calculated exactly and has been shown to be negligible.

The contributions to the radiative corrections from the diagrams of type a and from type b for $k_2 < k_{max}$ where k_{max}
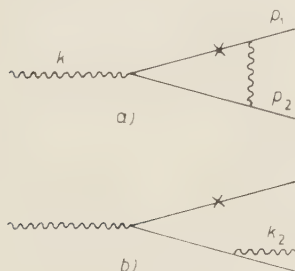

Fig. 1.

is the smallest value of the additional final state quantum k_2 which is detectable, give the total pair production cross section

$$\sigma = \sigma_0(1 + \delta) ,$$

---

[8] W. HEITLER: *The Quantum Theory of Radiation* (New York, 1954), third edition, p. 256.

where

$$\delta = \frac{e^2}{\pi} \left\{ 7.47 + 3.88 \ln (2k_{max}) \right\} =$$
$$= \left\{ 1.74 + 0.90 \ln (2k_{max}) \right\} \% .$$

σ_0 is the Bethe-Heitler cross section including the Coulomb correction of Davies, Bethe and Maximon [2]. k_{max} is here given in the system of reference in which the electron is at rest. In the present high energy approximation δ is independent of screening.

In the measurements of the absorbtion of photons also processes of type b with the emission of a high energy photon give contributions to δ of the order e^2. The differential cross section is again rather complicated [5]. The total cross section is being studied using methods analogous to the present calculations.

More detailed calculations together with spectral distributions will be given elsewhere.

* * *

We are greatly indebted to Dr. E. MALAMUD for information concerning his experiments.

Evolution in General Relativity.

W. Israel

Mathematics Department, University of Alberta - Edmonton

In a recent paper, Bonnor [1] has given some interesting examples of fluids in general relativity whose evolution in time is indeterminate. One of these examples suggests the possibility of a continuous transition from flat space-time to the expanding universe of Einstein-de Sitter.

We should like to make a few remarks concerning these examples, and we begin with a brief résumé of Bonnor's results.

Bonnor bases his models on certain special forms of the Robertson-Walker cosmological line-element

$$
(1) \qquad ds^2 = \frac{\exp[g(t)]}{(1 + k/4r^2)^2}(dx^2 + dy^2 + dz^2) - dt^2, \qquad r^2 = x^2 + y^2 + z^2, \qquad k = \text{const}.
$$

As is well known [2], the field equations $G_{ik} = -8\pi T_{ik}$ applied to (1) yield an energy tensor corresponding to a perfect fluid, with pressure p and energy density ϱ given by

$$
(2) \qquad 8\pi p = -k\exp[-g(t)] - \ddot{g} - \tfrac{3}{4}\dot{g}^2,
$$

$$
(3) \qquad 8\pi\varrho = 3k\exp[-g(t)] + \tfrac{3}{4}\dot{g}^2.
$$

Bonnor's general idea is to choose

$$
(4) \qquad g(t) = \begin{cases} g_1(t), & t \leqslant 0, \\ g_2(t), & t > 0, \end{cases}
$$

where $g_1(t)$, $g_2(t)$ are two different functions, analytic for $t \leqslant 0$ and $t > 0$ respectively, and satisfying

$$
g_1(0) = \lim_{t \to 0^+} g_2(t), \qquad g_1^{(n)}(0) = \lim_{t \to 0^+} g_2^{(n)}(t), \qquad (n = 1, 2, \ldots).
$$

[1] W. B. Bonnor: *Journ. Math. and Mech.*, **9**, 441 (1960).
[2] R. C. Tolman: *Relativity, Thermodynamics and Cosmology* (Oxford, 1934).

For his *first* model, Bonnor takes $k=0$ and

(5a) $$g(t) = 0 \,, \qquad\qquad t \leqslant 0,$$

(5b) $$g(t) = \tfrac{4}{3} \log \left(1 + at^m \exp \left[- b/t \right] \right) , \qquad t > 0.$$

The metrics associated with (5a), (5b) link up at $t=0$ with continuity of g_{ik}, p, ϱ and all their derivatives. The constants b, m can be chosen so that (1) approximates to the Einstein-de Sitter line-element (²) for large t. We are thus confronted with a model of space-time, flat and empty for negative t, which at $t=0$ « branches » smoothly into an expanding, matter-filled universe.

The model (5) leads to negative pressures for small, positive t, although (as Bonnor points out) the period of duration of negative pressure can be made arbitrarily small by adjustment of the constants. Bonnor gives a similar but more elaborate model which exhibits the same « branching » at $t=0$, and in addition satisfies

(6) $$0 < p < \tfrac{1}{3} \varrho \,.$$

How does the indeterminacy in these models arise? It should be noted first of all that the evolution of a perfect fluid is not completely fixed by the gravitational field equations alone; one additional equation is needed to make the problem determinate. For a simple fluid in isentropic flow, this equation takes the form

(7) $$S(p, \varrho) = \text{const} \,,$$

where S is entropy, and $S=S(p, \varrho)$ the equation of state. The equation of state implicit in a model of the type (4) is obtained by eliminating t between (2) and (3) and its form in general differs for $t < 0$ and for $t > 0$. However, from the physical point of view the equation of state is a characteristic relation whose form for a given fluid is *fixed once and for all*. *In general, therefore,* (4) *does not yield a physically satisfactory description of the evolution of a simple fluid.*

This objection is not applicable to a fluid mixture of two or more components which can interact with each other. In this case the system possesses additional internal degrees of freedom, the relation (7) is no longer valid, and without further information the evolution of the fluid is indeterminate. The « branching » of such a system at time $t=0$ can be attributed to an internal reaction beginning at this time. In the category of fluid mixtures we may include expanding cosmological models admitting creation of matter and negative pressures. According to an idea of PIRANI (³), these models can be thought of as interacting mixtures of matter and « gravitinos » (*i.e.* particles of zero rest-mass and negative energy).

The objection also does not apply to models, such as (5), which evolve from a *flat* space-time. The conditions $p=0$, $\varrho=0$ existing in such models for negative t, are clearly compatible with the equation of state derived from *any* line-element of the form (1) for $t > 0$, so long as that line-element tends to flatness when $t \to 0^+$.

Let us examine models of this latter type a little further. We have already noted that the model (5) does not satisfy the physically necessary conditions (6). This leads to the question: is there a Robertson-Walker cosmological model which

(³) F. A. E. PIRANI: *Proc. Roy. Soc.*, A **228**, 455 (1955).

evolves from a flat, empty space-time at $t=0$, and satisfies (6) for all positive t? So that our discussion should apply to models of the type considered by Bonnor, we shall not assume analyticity for $t=0$. Mathematically the problem can then be formulated as follows: is there a line-element of the form (1), analytic for $t > 0$, which satisfies (6) and the initial conditions

$$\lim_{t \to 0^+} g_{ik} = \eta_{ik}\,, \qquad \lim_{t \to 0^+} p = 0\,, \qquad \lim_{t \to 0^+} \varrho = 0\,?$$

(η_{ik} is the Minkowski tensor of flat space-time).

The condition of flatness as $t \to 0$ requires that $k=0$. Assuming $\varrho \neq 0$ almost everywhere in some open interval to the right of $t=0$ (*), the inequalities (6) are equivalent to

$$1 < \frac{\varrho + p}{\varrho} < \frac{4}{3}\ \text{p.p.}\,,$$

or, using (2) and (3),

$$(8) \qquad\qquad 1 < \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\dot g} < \frac{4}{3}\ \text{p.p.}\,.$$

But, by (3), $\dot g$ tends to zero with ϱ when $t \to 0$. This contradicts (8). Hence, *no matter-filled cosmological model satisfying (6) can evolve from a flat space-time.*

Finally, a remark on the general problem of predicting the evolution of a simple fluid, subject to (7), from given initial values of g_{ik}, $\partial g_{ik}/\partial x^r$, $V_i\,p$ and ϱ (cf. (^{4})). Granting the existence and analyticity of the solution to this Cauchy problem, it is not difficult to show that it is unique (apart from the freedom permitted by co-ordinate transformations) if the assigned values of ϱ and p on an initial space-like hypersurface satisfy

$$\varrho + p \neq 0\,.$$

If we exclude negative pressures as inadmissible, the only possibility of indeterminacy arises when $\varrho=0$, $p=0$ initially. This is the case we have already studied in connection with the special line element (1) (*).

To sum up, it appears that fluid-models of the type contemplated by Bonnor and satisfying normal physical requirements are only possible if there is an indeterminacy in the thermodynamical state of the system, as may happen, for instance, in the case of a mixture of two or more fluids whose mode of interaction is left arbitrary.

(*) The alternative possibility for a continuous function ϱ is that $\varrho \equiv 0$ is some interval $0 < t < t_1$. As we may easily verify from (2), (3) this leads to the trivial result that space-time is flat for $t > 0$.

(^{4}) Y. Foures-Bruhat: *Compt. Rend.*, **246**, 3319 (1958).

(*) If the form of the line-element is given *a priori* as in (1), the problem may yield a unique *analytic* solution for a simple fluid even when $\varrho = 0$, $p = 0$ initially. If $k = 0$, if $g(t)$, p, ϱ, vanish for $t = 0$ and if the equation of state $p = f(\varrho)$ is analytic then (2), (3) show that flat space-time is the only possible solution analytic for $t \geqslant 0$. This result is independent of the validity of (6).

The Fragmentation Probabilities
of Fast Heavy Cosmic-Ray Primaries in Teflon.

F. A. Brisbout, C. F. Gauld and C. B. A. McCusker (*)

F.B.S. Falkiner Nuclear Department, School of Physics,
University of Sydney - Sydney

1. – Introduction.

In order to extrapolate the charge spectrum of heavy primary cosmic rays to the top of the atmosphere, the fragmentation probabilities in air must be determined. Estimates of fragmentation probabilities in nuclear emulsions have already been made [1-6]. The question however arises as to whether these are the same for the light target nuclei (N, O) of the atmosphere. For this reason a sandwich stack of nuclear emulsions and Teflon $(C_2F_4)_n$ sheets was exposed and the fragmentations occur-ring in nuclear emulsion were compared with those taking place in the Teflon, which has an average Z very close to that of the atmosphere.

2. – Stack design.

The central portion of the stack con-sisted of 23 alternate sheets of Ilford G-5 emulsion (400 μm thick), and Teflon (500 μm thick). This was bounded on each side by 14 sheets of G-5 emulsion (600 μm thick). The purpose of the latter was to enable charge identifications to be made before the heavy primaries entered any photographically insensitive portion of the stack. Teflon $(CF_4)_n$ con-tains nuclei whose atomic weight is close to those of the chief elements of the atmosphere and is of sufficiently high density to make tracing back compar-atively easy.

3. – Scanning procedure.

The two outermost plates were area scanned, whilst the top edges of all the 600 μm emulsions were line scanned for

(*) Also supported by the Nuclear Research Foundation within the University of Sydney.
[1] P. H. Fowler, R. R. Hillier and C. J. Waddington: *Phil. Mag.*, **2**, 239 (1957).
[2] R. Cester, A. Debenedetti, C. M. Ga-relli, B. Quassiati, L. Tallone and M. Vigone: *Nuovo Cimento*, **7**, 371 (1958).
[3] M. Koshiba, G. Schultz and M. Schein: *Nuovo Cimento*, **9**, 1 (1958).
[4] W. Puschel: *Zeits. Naturfor.*, **13a**, 801 (1958).
[5] V. Y. Rayopadhye and C. J. Wad-dington: *Phil. Mag.*, **3**, 19 (1958).
[6] E. Lohrmann and M. Teucher: *Phys. Rev.*, **115**, 636 (1959).

heavy-primary tracks satisfying the fol-
lowing criteria:

  1) Blob density > 70 grains/100 μm.

  2) Length per plate > 2 mm.

4. – Charge identification.

Blob-density and δ-ray-density meas-
urements were made on each track and
a charge spectrum was obtained for the
point of entry into the sandwich section
of the stack. Measurements on every
5th track were remade to check for
consistency. Similar measurements by
the same observer were made on the
same tracks leaving the sandwich sec-
tions. Any observed differences were
investigated, thus locating the fragmen-
tation in the Teflon sheets. Readings of
different observers, and of an observer
at different times, were kept standard-
ized by frequent countings on known
« standard » tracks.

5. – Charge calibrabration.

The N_δ *versus* Z curve was cali-
brated using the reaction

$$B \to Li + \alpha ,$$

and the relationship

$$N = a + bZ^2 .$$

The first constant « a » was inde-
pendently determined by counting pseudo
δ rays along a bifilar hair line. A further
check was made against the reaction

$$C \to 3\alpha .$$

6. – Analysis of fragmentations.

In the case of fragmentations oc-
curring in Teflon, the expected position
of the heavy-primary track in the next
emulsion sheet was carefully scanned
until all the fragments of $Z > 2$ were
accounted for. This required scanning
an area sufficiently large to include at
least all α-particle fragments emitted
with an angle of 10^{-1} rad. Owing to
the relatively large path length which
each heavy primary must travel through
the stack, the heavy primary particles
selected were of sufficient energy to
satisfy this angular criterion.

7. – Results.

The results are given in Table I
which shows the « fragmentation prob-
abilities » for the production of various
types of secondary from various types
of primary for collisions in both Teflon
and emulsion. These numbers are not,
strictly, probabilities but rather the
average numbers of a given type of
particle produced in a given type of

TABLE I. – *Giving average numbers per interaction for various products and primaries for emulsion and Teflon.*

Type of interaction	Lα	LL	Mα	ML	MM	Hα	HL	HM	HH
Emulsion	$\frac{7}{18}=.39$	$\frac{3}{18}=.17$	$\frac{16}{17}=.94$	$\frac{3}{17}=.18$	$\frac{2}{17}=.12$	$\frac{6}{3}=2.0$	$\frac{0}{3}$	$\frac{1}{3}=.3$	$\frac{0}{3}$
Teflon	$\frac{12}{26}=.46$	$\frac{3}{26}=.12$	$\frac{24}{22}=1.09$	$\frac{5}{22}=.23$	$\frac{0}{22}$	$\frac{1}{2}=.5$	$\frac{1}{2}=.5$	$\frac{0}{2}$	$\frac{1}{2}=.5$

interaction. The usual definition of the terms L, M and H are used, namely that L nuclei are those with $3 \leqslant Z \leqslant 5$, M, those with $6 \leqslant Z \leqslant 9$, and H, those with $Z \geqslant 10$. The table gives the actual number of events and secondaries and their ratio as a decimal fraction.

8. – Discussion.

The results for emulsion in Table I may be compared with those of other workers summarized in the recent paper of LOHRMANN and TEUCHER [6]. It will be seen that within the limited statistical accuracy of this experiment they are in good agreement. The results for Teflon may be compared with those for « light »

target nuclei of the emulsion given in the same paper and again, within the statistical errors, there is good agreement. Thus, this experiment, using a material which is a very close approximation both in individual and average Z, to the atmosphere justifies the belief that « fragmentation probabilities » obtained from a study of collisions with $N_h < 7$ in emulsion are in close approximation to those occurring in the atmosphere.

* * *

Structure of Leptons and Mesons.

G. Rosen (*)

Institute for Theoretical Physics - Stockholm

This paper reports a quantum theory for the structure of leptons and mesons. The quantum theory is presented in axiomatic fashion without recourse to a classical Hamiltonian theory.

The state vector of a lepton or meson is written

$$(1) \qquad\qquad \psi = \psi_{\text{ext}} \psi_{\text{int}} \, ,$$

where ψ_{ext} and ψ_{int} are the *external* and *internal* states of the particle. The external state $\psi_{\text{ext}} = \psi_{\text{ext}}(\boldsymbol{x})$ can be represented by a set of complex-valued functions of the space-time co-ordinates $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$; ψ_{ext} satisfies a well-known wave equation for a lepton or meson of a given mass and spin. Similarly, the internal state $\psi_{\text{int}} = \psi_{\text{int}}(\boldsymbol{\xi})$ can be represented by a set of complex-valued functions of four real (dimensionless) internal co-ordinates $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$. The space-time co-ordinates $\boldsymbol{x}$ and the internal co-ordinates $\boldsymbol{\xi}$ are mutually independent variables. Not subject to space-time transformations and therefore Lorentz invariant scalars, the momenta conjugate to the ξ's define the structural observables of the particle:

$$
\begin{aligned}
&\textit{Lepton number} && \boldsymbol{N}_L = -i \frac{\partial}{\partial \xi_1} \, , \\[2mm]
&\textit{Strangeness number} && \boldsymbol{N}_S = -i \frac{\partial}{\partial \xi_2} \, , \\[2mm]
(2) \quad &\textit{Spin} && \boldsymbol{s} = -\frac{1}{2} \hbar \left(\frac{\partial}{\partial \xi_1} \right)^2 , \\[2mm]
&\textit{Electric charge} && \boldsymbol{q} = -ie \left(\frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_3} \right) , \\[2mm]
&\textit{Mass} && \boldsymbol{m} = m_{\pi^0} \left(\frac{\partial}{\partial \xi_4} \right)^4 .
\end{aligned}
$$

(*) National Science Foundation Postdoctoral Fellow.

A physically admissible ψ_{int} which is also a simultaneous eigenstate of the observables defined by eqs. (2) characterizes a single lepton or meson. Physically admissible internal states are determined by a subsidiary condition and a wave equation, relationships formulated in the following paragraphs.

1. – Subsidiary condition.

The subsidiary condition involves the internal co-ordinates ξ_1, ξ_2, ξ_3 symmetrically, and ξ_4 makes no appearance

$$(3) \qquad (\Delta' + 1)\psi_{\mathrm{int}} = 0 \,,$$

where

$$(4) \qquad \Delta' = W V^{-1} U \,,$$

$$(5a) \qquad U = \sum_{a=1}^{3} \xi_a \frac{\partial^2}{\partial \xi_a^2} \,.$$

$$(5b) \qquad V = \sum_{a=1}^{3} \xi_a^2 \frac{\partial}{\partial \xi_a} \,.$$

$$(5c) \qquad W = \sum_{a=1}^{3} \xi_a \frac{\partial}{\partial \xi_a} \,.$$

In eq. (4) V^{-1} denotes the formal inverse of the differential operator defined by eq. (5b).

There is a similarity between the subsidiary condition (3) and a Helmholtz equation, obtained if Δ' in eq. (3) were replaced by $\Delta = \sum_{a=1}^{3} \partial^2/\partial \xi_a^2$. Both Δ' and Δ have the same homogeneity with respect to (ξ_1, ξ_2, ξ_3); this fact is expressed by the commutation relations

$$(6) \qquad [W, \Delta'] = -2\Delta' \,, \qquad [W, \Delta] = -2\Delta \,,$$

in which W is defined by eq. (5c). Moreover, for a function f which is analytic in ξ_1, ξ_2, ξ_3 in the neighborhood of the origin, it follows that

$$(7) \qquad \Delta' f = W V^{-1} U \left((f)_{\xi=0} + \sum_{a=1}^{3} \left(\frac{\partial f}{\partial \xi_a} \right)_{\xi=0} \xi_a + \frac{1}{2} \sum_{a,b=1}^{3} \left(\frac{\partial^2 f}{\partial \xi_a \partial \xi_b} \right)_{\xi=0} \xi_a \xi_b + \dots \right) =$$

$$= W V^{-1} \left(\sum_{a=1}^{3} \left(\frac{\partial^2 f}{\partial \xi_a^2} \right)_{\xi=0} \xi_a + \dots \right) = W \left(\sum_{a=1}^{3} \left(\frac{\partial^2 f}{\partial \xi_a^2} \right)_{\xi=0} \ln \xi_a + \dots \right) = \sum_{a=1}^{3} \left(\frac{\partial^2 f}{\partial \xi_a^2} \right)_{\xi=0} + \dots \,.$$

Thus, Δ' acts like Δ in the neighborhood of the origin,

$$(8) \qquad (\Delta' f)_{\xi=0} = (\Delta f)_{\xi=0} \,.$$

The subsidiary condition (3) assumes a more tractable form with the elimination of the integration operator V^{-1}. Applying the operator V to eq. (3) and using the

commutation relation derived from eqs. (5b) and (5c), we find that

(9) $$(WU - U + V)\psi_{\text{int}} = 0 .$$

Finally, by putting eqs. (5) into eq. (9), we obtain the subsidiary condition in its most useful form

(10) $$\left(\sum_{a,b=1}^{3} \xi_a \xi_b \frac{\partial^3}{\partial \xi_a \partial \xi_b^2} + \sum_{a=1}^{3} \xi_a^2 \frac{\partial}{\partial \xi_a} \right) \psi_{\text{int}} = 0 .$$

2. – Wave equation.

The wave equation is

(11) $$\left(\sum_{\mu=1}^{4} \Lambda_\mu \frac{\partial^2}{\partial \xi_\mu^2} + 1 \right) \psi_{\text{int}} = 0 ,$$

in which the four Λ's are matrices. An algebra for the Λ's is defined partially by the system of equations

(12) $$\begin{cases} [\Lambda_1, \Lambda_2] = 0 , & (\Lambda_2)^2 = 0 , \\ \{\Lambda_3, \Lambda_4\} = 0 , & (\Lambda_4)^2 = 1 , \\ [\Lambda_1, [\Lambda_4, \Lambda_2]] = \Lambda_2 , \end{cases}$$

in which square brackets denote commutators and curly brackets denote an anti-commutator. In order to complete a specification of the Λ-algebra, it is convenient to introduce an auxiliary matrix Λ_0; the latter matrix squares to zero

(13) $$(\Lambda_0)^2 = 0 ,$$

and has the following anticommutators with the original Λ's:

(14) $$\begin{cases} \frac{1}{2}\{\Lambda_0, \Lambda_1\} = \Lambda_0 , & \{\Lambda_0, \Lambda_2\} = 0 , \\ \{\Lambda_0, \Lambda_3\} = 0 , & \frac{1}{2}\{\Lambda_0, \Lambda_4\} = \alpha . \end{cases}$$

The fine structure constant $\alpha = e^2/\hbar c = (137.04)^{-1}$ enters the theory in a fundamental way, fixing the anticommutator $\{\Lambda_0, \Lambda_4\}$. Finally, the Λ-algebra is closed by specifying the commutation relations

(15) $$\begin{cases} i[\Lambda_1, \Lambda_3] = \frac{1}{2}(\Lambda_0 + \Lambda_2) + \Lambda_1 - 1 , \\ i[\Lambda_2, \Lambda_3] = \Lambda_0 + \Lambda_1 - 1 . \end{cases}$$

Note that the wave eq. (11) and the Λ-algebra, as well as the subsidiary condition (3), are preserved under the transformation

(16) $\qquad\qquad\qquad \psi_{\text{int}} \to S\psi_{\text{int}}\,, \qquad\qquad \Lambda_\mu \to S\Lambda_\mu S^{-1}\,, \qquad\qquad [\mu = 0 \text{ to } 4],$

where S is a non-singular matrix of constant elements.

There is no one-dimensional representation of the Λ-algebra. The two-dimensional representations are deduced by working first with eqs. (12), (13) and (14). To within an S-transformation (16), straightforward analysis shows that (*)

(17)
$$\begin{cases} \Lambda_0 = \begin{pmatrix} 0 & 0 \\ 2\alpha & 0 \end{pmatrix}, & \Lambda_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, \\[3mm] \Lambda_2 = \begin{pmatrix} 0 & 0 \\ \lambda_2 & 0 \end{pmatrix}, & \Lambda_3 = \begin{pmatrix} -i\lambda_3 & 0 \\ 0 & i\lambda_3 \end{pmatrix}, & \Lambda_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{cases}$$

in which λ_2 and λ_3 are arbitrary complex numbers. Substituting the representation (17) into eqs. (15) produces the conditions

(18) $\qquad\qquad\qquad \lambda_3 = \frac{1}{2} + \frac{1}{2}\lambda_2 + \alpha\,, \qquad\qquad 2\lambda_2\lambda_3 = \frac{1}{2} + 2\alpha\,,$

with the solutions

(19) $\qquad \lambda_2 = -(\frac{1}{2} + \alpha) \pm \sqrt{\frac{3}{4} + 3\alpha + \alpha^2}\,, \qquad \lambda_3 = \frac{1}{2}(\frac{1}{2} + \alpha) \pm \frac{1}{2}\sqrt{\frac{3}{4} + 3\alpha + \alpha^2}\,.$

The lepton-meson representation of the Λ-algebra is characterized by the trace conditions

(20) $\qquad\qquad\qquad \text{Tr}\,(\Lambda_1) = 2\,, \qquad\qquad \text{Tr}\,(\Lambda_2\Lambda_4) < 0\,,$

and obtained from eqs. (17) and (19) by taking the negative square-roots in eq. (19), so that

(21) $\qquad\qquad\qquad \lambda_2 = -1.3859\,, \qquad\qquad \lambda_3 = -.18566\,.$

Other representations of the Λ-algebra will be discussed in a subsequent paper.

The wave equation is stated in a useful form by substituting the matrices of eqs. (17) into eq. (11)

(22)
$$\begin{pmatrix} \dfrac{\partial^2}{\partial\xi_1^2} - i\lambda_3\dfrac{\partial^2}{\partial\xi_3^2} + 1 & \dfrac{\partial^2}{\partial\xi_4^2} \\[4mm] \dfrac{1}{2}\dfrac{\partial^2}{\partial\xi_1^2} + \lambda_2\dfrac{\partial^2}{\partial\xi_2^2} + \dfrac{\partial^2}{\partial\xi_4^2} & \dfrac{\partial^2}{\partial\xi_1^2} + i\lambda_3\dfrac{\partial^2}{\partial\xi_3^2} + 1 \end{pmatrix} \psi_{\text{int}} = 0\,.$$

(*) A trivial representation, for which $\Lambda_2 = 0$, is ignored.

3. – Lepton-meson states.

Solutions to the subsidiary condition (10) and the wave eq. (22) can be written as a linear superposition of simple waves

$$(23) \qquad \psi_{\text{int}} = (\text{const}) \exp \Big[i \sum_{\mu=1}^{4} N_{\mu} \xi_{\mu} \Big] ,$$

in which the four N's are constants. The subsidiary condition (10) applied to eq. (23) requires that

$$(24) \qquad \sum_{a=1}^{3} \xi_a N_a \sum_{b=1}^{3} \xi_b N_b^2 - \sum_{a=1}^{3} \xi_a^2 N_a = 0 .$$

This relation must be satisfied identically for all values of ξ_1, ξ_2, ξ_3, and thus we have

$$(25) \qquad \tfrac{1}{2}(N_a N_b^2 + N_a^2 N_b) - N_a \delta_{ab} = 0 , \qquad\qquad [a, b = 1, 2, 3].$$

The diagonal part of eq. (25) restricts each N_a to the values $+1$, -1, or 0. The off-diagonal part of eq. (25) excludes solutions with two of the N_a equal to $+1$ or with two of the N_a equal to -1. All solutions to eq. (25) are listed in the Table.

By substituting the form (23) into the wave eq. (22), we find the determinantal condition

$$(26) \qquad \begin{vmatrix} -N_1^2 + i\lambda_3 N_3^2 + 1 & -N_4^2 \\ -\tfrac{1}{2} N_1^2 - \lambda_2 N_2^2 - N_4^2 & -N_1^2 - i\lambda_3 N_3^2 + 1 \end{vmatrix} = 0 ,$$

which reduces to

$$(27) \qquad N_4^2 = -\Big(\frac{1}{4} N_1^2 + \frac{\lambda_2}{2} N_2^2 \Big) + \sqrt{ \Big(\frac{1}{4} N_1^2 + \frac{\lambda_2}{2} N_2^2 \Big)^2 + (N_1^2 - 1)^2 + \lambda_3^2 N_3^4 } .$$

Here we have taken the positive square-root, for if the solution stated by eq. (23) is to remain bounded as $\xi_4 \to \pm\infty$, N_4^2 must be non-negative. A special case arises if $N_1^2 = N_2^2 = 1$ and $N_3^2 = 0$, for then the negative root is also admissible, and we have the extra solution

$$(28) \qquad N_4^2 = 0 \qquad\qquad [\text{for } N_1^2 = N_2^2 = 1, N_3^2 = 0].$$

Values of N_4^2 associated with each set of admissible (N_1, N_2, N_3) are listed in the Table.

The eigenvalues of spin, electric charge, and mass, defined according to eqs. (2), fill the remaining columns of the Table. Each row in the Table is labeled with the appropriate lepton or meson symbol. The muon is assigned a strangeness number ± 1. Also note that the extra solution in eq. (28) (in parentheses in the Table) admits a massless muon. No other undiscovered lepton or meson is predicted by the theory, and the theoretical eigenvalues for the well-established par-

ticles agree with experiment. The theoretical mass spectrum is certainly accurate to within $\frac{1}{2}$ of 1%. More refined experiments will determine the precision of the theory.

Table of lepton-meson states.

Solution label	N_L $(=N_1)$	N_s $(=N_2)$	N_3	N_4^2	$s/\hbar$ $(=\frac{1}{2}N_1^2)$	q/e $(=N_2+N_3)$	m/m_{π^0} $(=N_4^4)$	m/m_e $(=(.003771)^{-1}$ $m/m_{\pi^0})$
ν^0	$+1$	0	0	0	$\frac{1}{2}$	0	0	0
$\bar{\nu}^0$	-1	0	0			0		
e^-	$+1$	0	-1	$.06141$	$\frac{1}{2}$	-1	$.003771$	1
e^+	-1	0	$+1$			$+1$		
μ^-	$+1$	-1	0	$.8859$	$\frac{1}{2}$	-1	$.7848$	208.1
μ^+	-1	$+1$	0	(0)		$+1$	(0)	(0)
π^0	0	0	0	1	0	0	1	265.2
π^+	0	0	$+1$	1.0171	0	$+1$	1.0345	274.3
π^-	0	0	-1			-1		
τ^+	0	$+1$	0	1.9096	0	$+1$	3.6466	967.1
τ^-	0	-1	0			-1		
θ^0	0	$+1$	-1	1.9237	0	0	3.7005	981.4
$\bar{\theta}^0$	0	-1	$+1$			0		

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